Beginning with the classical recursions for polynomial interpolation by Aitken and Neville we consider their algebraic fundamentals and some of their generalizations. The discussed topics primarily originate from scientific contributions by Mariano Gasca.

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The president of the organizing committee, Prof. Francisco-Javier Muñoz-Delgado, has asked me to present at this conference a short plenary talk speaking about subjects of Mariano’s and mine interest. One topic lies in the center of our common interests. That are the classical Aitken-Neville formulae, their algebraic fundamentals and their generalizations. I will talk about results obtained in the last 30 years mainly by Mariano Gasca and his coworkers.

Let me begin with the classical linear interpolation problem. I’ll use notations of one of Mariano’s recent papers [19]. Let $D \subset \mathbb{R}^d$ and $X \subset D$ be a finite subset, let $\mathbb{P}$ be a finite dimensional linear space of real valued functions on $D$. By $E_{X,\mathbb{P}}$ we denote the restriction map

$$E_{X,\mathbb{P}} : \mathbb{P} \ni p \mapsto p|_X = (p(x) : x \in X) \in \mathbb{R}^X.$$ 

The **Lagrange interpolation problem on** $X$ is:

(1) given $f \in \mathbb{R}^X$, find $p \in \mathbb{P}$ such that $E_{X,\mathbb{P}}(p) = f$.

A set $X$ is called **$\mathbb{P}$-correct** iff the Lagrange interpolation problem on $X$ has a unique solution for every function $f \in \mathbb{R}^X$. In this case the interpolant of $f$ is denoted by
$E^{-1}_{X,P}(f) = p(\cdot, X, P, f)$ and the interpolation remainder by $r(\cdot, X, P, f) := f - p(\cdot, X, P, f)$. Necessarily, $\dim P = \#X$.

Assume $\dim P = n = \#X$. By choosing a basis $(p_1, \ldots, p_n)$ of $P$ the requirement that $p$ and $f$ agree on $X$ constitutes a system of $n$ linear equations for $n$ unknowns (the coefficients of the basic elements) which is nonsingular for every $f$ iff $X$ is $P$-correct.

Well known is the classic Aitken-Neville recursion. If $\mathbb{P}_n = \mathbb{P}_n(\mathbb{R}^1) = \text{span}\{\pi_0, \ldots, \pi_{n-1}\}$, $\pi_j(x) = x^j$, is the space of algebraic polynomials of order $n$ (degree $n - 1$) in one real variable, $X = \{x_1, \ldots, x_n\} \subset D \subset \mathbb{R}^1$ is a finite set of cardinality $n$ then for all $x$

$$(2) \quad p(x, X, \mathbb{P}_n, f) = \frac{x_n - x}{x_n - x_1} p(x, \{x_1, \ldots, x_{n-1}\}, \mathbb{P}_{n-1}, f) + \frac{x - x_1}{x_n - x_1} p(x, \{x_2, \ldots, x_n\}, \mathbb{P}_{n-1}, f).$$

Here it is well known that every finite set $Y \subset \mathbb{R}^1$ of cardinality $m$ is $\mathbb{P}_m$-correct for every $m = 1, 2, \ldots$. Hence the recurrence formula (2) can be applied iteratively starting with the initializations $p(x, \{x_k\}, \mathbb{P}_1, f) = f(x_k)$, $k = 1, \ldots, n$. I am sure that everyone knows the simple two line proof of formula (2).

In the next step we are going to observe that the Aitken-Neville weights can be expressed by two special interpolation remainders. The fundamental theorem of algebra yields

$$r_1 = r(x, \{x_1, \ldots, x_{n-1}\}, \mathbb{P}_{n-1}, \pi_{n-1}) = (x - x_1) \cdots (x - x_{n-1})$$
$$r_2 = r(x, \{x_2, \ldots, x_n\}, \mathbb{P}_{n-1}, \pi_{n-1}) = (x - x_2) \cdots (x - x_n).$$

Subtraction gives $r_1 - r_2 = (x - x_2) \cdots (x - x_{n-1}) \cdot [(x - x_1) - (x - x_n)] = (x - x_2) \cdots (x - x_{n-1})(x_n - x_1)$ and for all $x \notin \{x_2, \ldots, x_{n-1}\}$ the Aitken-Neville weights allow the representation

$$\frac{x_n - x}{x_n - x_1} = \frac{r_2}{r_2 - r_1}, \quad \frac{x - x_1}{x_n - x_1} = \frac{-r_1}{r_2 - r_1}.$$

If $\mathbb{P}_n = \text{span}\{p_1, \ldots, p_n\}$ is a complete Čebyšev-system on $D$ (CT-system) defined by the property that every finite set $Y \subset D$ of cardinality $m$ is $\mathbb{P}_m$-correct ($m = 1, \ldots, n$), then it should be expected that the Aitken-Neville formula holds [32]

$$(3) \quad p(x, \{x_1, \ldots, x_n\}, \mathbb{P}_n, f) = r_2(x) \cdot p(x, \{x_1, \ldots, x_{n-1}\}, \mathbb{P}_{n-1}, f) - r_1(x) \cdot p(x, \{x_2, \ldots, x_n\}, \mathbb{P}_{n-1}, f)$$
$$r_2(x) = r(x, \{x_2, \ldots, x_n\}, \mathbb{P}_{n-1}, p_n), \quad r_1(x) = r(x, \{x_1, \ldots, x_{n-1}\}, \mathbb{P}_{n-1}, p_n)$$

with initializations $p(x, \{x_k\}, \mathbb{P}_1, f) = \frac{f(x_k)}{p_1(x_k)} p_1(x)$, $k = 1, \ldots, n$. 

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The interpolation problem written as a system of linear equations for the coefficients $c_k = c_k(\{x_1, \ldots, x_n\}, \mathbb{P}_n, f)$ of the interpolant

$$y = p(x, \{x_1, \ldots, x_n\}, \mathbb{P}_n, f) =: \sum_{k=1}^{n} c_k p_k(x)$$

reads with the notations $\langle L, u \rangle := u(x)$, $\langle L, u \rangle := u(x)$

$$
\begin{pmatrix}
\langle L_1, p_1 \rangle & \ldots & \langle L_1, p_{n-1} \rangle & \langle L_1, p_n \rangle & 0 \\
\langle L_2, p_1 \rangle & \ldots & \langle L_2, p_{n-1} \rangle & \langle L_2, p_n \rangle & 0 \\
\vdots & & \vdots & \vdots & \vdots \\
\langle L_{n-1}, p_1 \rangle & \ldots & \langle L_{n-1}, p_{n-1} \rangle & \langle L_{n-1}, p_n \rangle & 0 \\
\langle L_n, p_1 \rangle & \ldots & \langle L_n, p_{n-1} \rangle & \langle L_n, p_n \rangle & 0 \\
-(\langle L_1, p_1 \rangle) & -\ldots & -(\langle L_1, p_{n-1} \rangle) & -(\langle L_1, p_n \rangle) & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{n-1} \\
c_n
\end{pmatrix}
= 
\begin{pmatrix}
\langle L_1, f \rangle \\
\langle L_2, f \rangle \\
\vdots \\
\langle L_{n-1}, f \rangle \\
\langle L_n, f \rangle
\end{pmatrix}
$$

Here we have bordered the system for the coefficients by equation (4) thus introducing $y$ as another unknown. This $(n + 1) \times (n + 1)$ system has to be solved for $y$.

By making use of the notations chosen actually we have generalized the interpolation problem to the problem of linear extrapolation by allowing $\mathbb{P}_n = \text{span}\{p_1, \ldots, p_n\}$ a linear space of dimension $n$ and $L, L_1, \ldots, L_n \in \mathbb{P}_n^*$ linear functionals on $\mathbb{P}_n$. What we are looking for is now the value $y = \langle L, p \rangle$ of the extrapolant $p = p(\{L_1, \ldots, L_n\}, \mathbb{P}_n, f) = \sum_{k=1}^{n} c_k \cdot p_k$, defined by

$$C_n \cdot c = b, \quad C_n = (\langle L_i, p_j \rangle)_{i=1,\ldots,n}^{j=1,\ldots,n}, \quad b = (\langle L_i, f \rangle)_{i=1,\ldots,n}, \quad c = (c_j)_{j=1,\ldots,n}$$

where $f$ is any element in ran $\mathbb{P}_n^*$ (such that $\langle L_i, f \rangle$ $(i = 1, \ldots, n)$ are defined). Observe that $y$ is the Schur complement of the matrix

$$C_n \text{ in } C_{n+1} = (\langle L_i, p_j \rangle)_{i=1,\ldots,n}^{j=1,\ldots,n+1}$$

$$y = \frac{\det C_{n+1}}{\det C_n},$$

where we use the notations $p_{n+1} := f$, $L_{n+1} := L$ and $\langle L_{n+1}, p_{n+1} \rangle := 0$.

In the spirit of Aitken and Neville actually we want to compute $y$ recursively from the solutions of some smaller dimensional extrapolation problems. To get an idea how elimination techniques can be used let us derive (3) from (5). We consider two submatrices of the coefficient matrix of (5) $A_1$ having a solid red frame and $A_2$ having a solid green frame

$$A_1 = (\langle L_i, p_j \rangle)_{i=1,\ldots,n}^{j=1,\ldots,n-1}, \quad A_2 = (\langle L_i, p_j \rangle)_{i=2,\ldots,n}^{j=1,\ldots,n-1}.$$
We assume that they are nonsingular. By block elimination of $c_1, \ldots, c_{n-1}$ using $A_1$ respectively $A_2$ as pivot the system (5) is transformed into a system of 2 equations for 2 unknowns $c_n, y$ which read

\begin{align}
& r_1 \cdot c_n + y = w_1 \\
& r_2 \cdot c_n + y = w_2.
\end{align}

We see that $w_1$ is the Schur complement of $A_1$ in the matrix $B_1$ which in (5) is framed by a solid and an interrupted red frame hence

\[ w_1 = A_1^{-1} \cdot \begin{pmatrix} \langle L_1, f \rangle \\ \vdots \\ \langle L_{n-1}, f \rangle \end{pmatrix} \cdot \langle L, p_1, \ldots, L, p_{n-1} \rangle = \langle L, p(\{L_1, \ldots, L_{n-1}\}, \mathbb{P}_{n-1}, f) \rangle \]

which in terms of interpolation is $p(x, \{x_1, \ldots, x_{n-1}\}, \mathbb{P}_{n-1}, f)$. Similarly, one has

\[ w_2 = \langle L, p(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, f) \rangle \]

\[ r_1 = -\langle L, r(\{L_1, \ldots, L_{n-1}\}, \mathbb{P}_{n-1}, p_n) \rangle \]

\[ r_2 = -\langle L, r(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, p_n) \rangle. \]

Clearly, if $r_1 \neq r_2$, then the matrix $C_n$ is regular, and by a final elimination step or directly by Cramer’s rule we get

\[ y = \frac{r_2 \cdot w_1 - r_1 \cdot w_2}{r_2 - r_1}. \]

This is formula (3) in a more general form. We remark that for this general Aitken-Neville recursion to hold we need the condition $r_1 \neq r_2$. It is not hard to see that it is equivalent with $L, L_1, \ldots, L_{n-1}$ linearly independent on $\mathbb{P}_{n-1}$ [33]. Clearly, in case of interpolation by a CT-system this condition holds.

I think it is worthwhile to show that there is a really short proof of (10) almost as simple as that of its classical original. Using the duality of $\mathbb{P}_n$ and $\mathbb{P}_n^*$ under the brackets $\langle \cdot, \cdot \rangle$, $y$ is also the value at $f$ of the linear combination $q$ of the functionals $L_1, \ldots, L_n$ that interpolates the functional $L$ with respect to $p_1, \ldots, p_n$: $\langle q, f \rangle = \langle L, p \rangle$ where

\[ q = q(\{p_1, \ldots, p_n\}, \text{span}\{L_1, \ldots, L_n\}, L) =: \sum_{j=1}^n a_j \cdot L_j \]

is defined by $q|_{\mathbb{P}_n} = L|_{\mathbb{P}_n}$. In fact, $\langle q, f \rangle$ being the Schur complement of $C_n^T$ in $C_{n+1}^T$ is the same as the Schur complement of $C_n$ in $C_{n+1}$ which is $y = \langle L, p \rangle$. Denote the right hand
side of (10) by \( \langle Q, f \rangle \). Since

\[ w_2 = \langle L, p(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, f) \rangle = \langle q(\{p_1, \ldots, p_{n-1}\}, \text{span} \{L_2, \ldots, L_n\}, L), f \rangle \]

and

\[ w_1 = \langle q(\{p_1, \ldots, p_{n-1}\}, \text{span} \{L_1, \ldots, L_{n-1}\}, L), f \rangle \]

for fixed \( L \) such that \( L, L_2, \ldots, L_{n-1} \) are linearly independent on \( \mathbb{P}_{n-1} \) we have \( Q \in \mathbb{P}_n^* \) is a linear combination of the functionals \( L_1, \ldots, L_n \) since the Aitken-Neville weights are numbers adding to one. It is easily checked that

\[ \langle Q, p_j \rangle = \langle L, p_j \rangle \]

for \( j = 1, \ldots, n \). By the unicity of the extrapolant we infer

\[ Q = q \]

which, when applied to any \( f \in \mathbb{P}_n^* \), gives

\[ \langle Q, f \rangle = \langle q, f \rangle = \langle L, p \rangle \]

Again, the condition \( L, L_2, \ldots, L_{n-1} \) linearly independent on \( \mathbb{P}_{n-1} \) is crucial.

In the 1980’s we, that means Mariano and me, have generalized this approach ([2],[3],[4],[11]) by making use of so called elimination strategies (e.s.). Instead of giving their definition (which is somewhat technical) I will consider here only the two most important examples already used in [1]. Let \( I = (1,2,\ldots,n) \) be an index list, \( 1 \leq k \leq n \) and \( m := n - k + 1 \). The family \( G = (G_s, G_s^0)_{s=1,\ldots,m} \) of sublists of \( I \)

\[ G_s = G_1^0 = (1,\ldots,k-1), \quad G_s = G_s^0 \cup (s + k - 1), \quad s = 1,\ldots,m \]

we have called the special Gauss \((k,m)\)-elimination strategy over \( I \). Another \((k,m)\)-elimination strategy over \( I \) is

\[ N_s^0 = (s, s + 1, \ldots, s + k - 2), \quad N_s := N_s^0 \cup (s + k - 1), \quad s = 1,\ldots,m. \]

It has been called the Neville \((k,m)\)-elimination strategy over \( I \). Using elimination strategies we have derived generalized Schur complements and we have got a generalization of Sylvester’s identity on determinants [1],[2].

**Theorem 1.** Let \( A = A(I) \in K^{n \times n} \) be any \( n \times n \)-matrix over a field \( K \) of characteristic zero. Let \( \Sigma = (I_s, I_s^0)_{s=1,\ldots,m} \) and \( K = (K_s, K_s^0)_{s=1,\ldots,m} \) be two \((k,m)\)-elimination strategies over \( I \), one being Gaussian. Then

\[ \det \left( \det A \left( \begin{array}{c} K_s^0 \\ I_s^0 \end{array} \right) \right)_{s=1,\ldots,m} = \sigma \cdot \det A \prod_{s=2}^m \det A \left( \begin{array}{c} K_s^0 \\ I_s^0 \end{array} \right) \]

with a sign factor \( \sigma \) depending only on \( \Sigma \) and \( K \).

Let us have now a second look at the linear system (5) corresponding to the linear extrapolation problem. It can be written in the form

\[ \begin{pmatrix} C_n & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} c \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \]
where $b, c$ and $C_n$ were defined in (6) and $v = (-\langle L, p_1 \rangle, \ldots, -\langle L, p_n \rangle)$. Denoting the matrix of (12) by $B$ an application of Cramer’s rule yields

$$y = \frac{\det C_{n+1}}{\det B} = \frac{\det C_{n+1}}{\det C_n}$$

with $C_{n+1}$ defined as above. Obviously, this ”calls” for application of Sylvester’s generalized identity with elimination strategies being the same for numerator and denominator. Then the sign factor cancels. For instance, we will take the special $(n, 2)$-Gauss elimination strategy over $I = (1, \ldots, n + 1)$

$$G_1 = (1, \ldots, n - 1, n), \quad G_1^0 = G_2^0 = (1, \ldots, n - 1), \quad G_2 = (1, \ldots, n - 1, n + 1)$$

and the special $(n, 2)$-Neville elimination strategy over $I$

$$N_1 = (1, \ldots, n - 1, n), \quad N_1^0 = G_1^0, \quad N_2^0 = (2, \ldots, n - 1, n + 1), \quad N_2 = (2, \ldots, n, n + 1).$$

Assume $\det B \left( \begin{array}{c} G_2^0 \\ N_2^0 \end{array} \right) \neq 0$ i.e. $L, L_2, \ldots, L_{n-1}$ linearly independent on $\mathbb{P}_{n-1}$. Then according to Theorem 1

$$\det B = \det C_n \neq 0 \iff \det \left( \begin{array}{cc} \det B \left( \begin{array}{c} G_1 \\ N_1 \end{array} \right) & \det B \left( \begin{array}{c} G_2 \\ N_1 \end{array} \right) \\ \det B \left( \begin{array}{c} G_1 \\ N_2 \end{array} \right) & \det B \left( \begin{array}{c} G_2 \\ N_2 \end{array} \right) \end{array} \right) \neq 0.$$ 

In this case

$$y = \frac{\det \left( \begin{array}{cc} \det C_{n+1} \left( \begin{array}{c} G_1 \\ N_1 \end{array} \right) & \det C_{n+1} \left( \begin{array}{c} G_2 \\ N_1 \end{array} \right) \\ \det C_{n+1} \left( \begin{array}{c} G_1 \\ N_2 \end{array} \right) & \det C_{n+1} \left( \begin{array}{c} G_2 \\ N_2 \end{array} \right) \end{array} \right)}{\det \left( \begin{array}{cc} \det B \left( \begin{array}{c} G_1 \\ N_1 \end{array} \right) & \det B \left( \begin{array}{c} G_2 \\ N_1 \end{array} \right) \\ \det B \left( \begin{array}{c} G_1 \\ N_2 \end{array} \right) & \det B \left( \begin{array}{c} G_2 \\ N_2 \end{array} \right) \end{array} \right)}.$$

Dividing the first row of both determinants in the numerator and denominator by

$$\det B \left( \begin{array}{c} G_1^0 \\ N_1^0 \end{array} \right) = \det C_{n+1} \left( \begin{array}{c} G_1^0 \\ N_1^0 \end{array} \right) \neq 0$$

and the second by

$$\det B \left( \begin{array}{c} G_1^0 \\ 2, \ldots, n \end{array} \right) = \det C_{n+1} \left( \begin{array}{c} G_1^0 \\ 2, \ldots, n \end{array} \right) \neq 0$$
which are assumed to be distinct from zero (that means we assume that the lower dimensional extrapolants exist) we get

\[ y = \frac{\det \left( \begin{array}{c}
\langle L, r(\{L_1, \ldots, L_{n-1}\}, \mathbb{P}_{n-1}, p_n)\rangle \\
\langle L, r(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, p_n)\rangle
\end{array} \right) \langle L, p(\{L_1, \ldots, L_{n-1}\}, \mathbb{P}_{n-1}, f)\rangle}{\det \left( \begin{array}{c}
\langle L, r(\{L_1, \ldots, L_{n-1}\}, \mathbb{P}_{n-1}, p_n)\rangle \\
\langle L, r(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, p_n)\rangle
\end{array} \right) \langle L, p(\{L_2, \ldots, L_n\}, \mathbb{P}_{n-1}, f)\rangle} \]

i.e. formula (10). This approach to the linear extrapolation problem in more generality is treated in [34].

Generalized Schur complements have found applications in matrix theory, in particular for totally positive (TP) and strictly totally positive (STP) matrices. They are defined by the property that all their minors are nonnegative resp. strictly positive. It is easy to see that for a STP-matrix all generalized Schur complements with respect to the Neville e.s. are STP. Using this fact we (Mariano and me) in 1987 [3],[4] have derived a test for \( n \times n \)-matrices to be STP of complexity \( O(n^4) \). In the following years Mariano and Juan Manuel Peña have studied Neville e.s. exhaustively. In 1992 in [5]-[10] they have obtained a test for STP of complexity \( O(n^3) \). Moreover, they have got a deep insight in the structure of TP and STP matrices. Let me mention just one remarkable characterization of STP matrices: A nonsingular \( n \times n \) matrix \( A \) is STP iff

\[ A = F_{n-1} \cdots F_1 \cdot D \cdot G_1 \cdots G_{n-1} \]

where for \( i = 1, \ldots, n-1 \) \( F_i \) is a bidiagonal, lower triangular matrix of the form

\[ F_i = \begin{pmatrix}
1 & 0 & 1 \\
& \ddots & \ddots \\
& 0 & 1 \\
& & + 1 \\
& & & + 1 \\
& & & & \ddots \\
& & & & & + 1
\end{pmatrix} \]

where free places are occupied with zeros and a symbol + stands for a positive element and the first + stands in row \( i+1 \). \( D \) is a diagonal matrix with positive diagonal elements and \( G_i \) has the transposed form of \( F_i \). They have obtained similar results for TP and sign regular matrices and made interesting applications of Neville elimination to CAGD.
Another topic which Mariano and me are interested in is interpolation by polynomials of several variables. It is well known that in $\mathbb{R}^d$, $d > 1$, only for particular knot sets interpolating multivariate polynomials do exist, and only for even more particular knot systems Aitken-Neville recursions are known.

Mariano has started his research in this field in the early 1980’s [20]. His paper [12] of 1987 together with E. Lebrón has proven fundamental for Aitken-Neville formulae for multivariate interpolation. In papers [14]-[17] we (Mariano and me) have used their general result to establish Aitken-Neville recursions for rather simple knot configurations such as geometric or regular meshes or tensor product grids. For such configurations we have computed the Aitken-Neville weights explicitly. By making use of projective mappings of $\mathbb{R}^d$ we have been able to get explicit weights for Aitken-Neville recursions on $(d+1)$-pencil lattices. Later, in a fruitful cooperation with his former pupil and now colleague Jesus Carnicer and with Tomas Sauer (University of Giessen, Germany) in a long series of papers (see references [14]-[26]) they have obtained remarkable progress in describing knot configurations which allow Aitken-Neville recursions in multivariate polynomial interpolation.

To give only one example we consider polynomial interpolation at knots forming a principal lattice of degree $n$. Consider multiindices $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}_{d+1}$ with norm $|\alpha| = \sum_{j=0}^{d} \alpha_j$. By $\varepsilon_j \in \mathbb{N}_{d+1}$ we denote the $j$-th unit vector.

**Definition 2.** Let $Y = \{y_0, \ldots, y_d\} \subset \mathbb{R}^d$ be affinely independent, i.e. these points are vertices of a nondegenerate simplex in $\mathbb{R}^d$. A principal lattice of degree $n$ generated by $Y$ is the set of points

$$X = \{x_{\alpha} = \sum_{j=0}^{d} \frac{\alpha_j}{n} \cdot y_j : |\alpha| = n\}.$$

The $j$-th barycentric coordinate of $x$ with respect to $[Y]$ is

$$\lambda_{j,Y}(x) := \frac{\text{vol} \ [y_0, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_d]}{\text{vol} \ [y_0, \ldots, y_d]} \in \mathbb{P}_1(\mathbb{R}^d)$$

where $[Y]$ denotes the convex hull of $Y$. From the fundamental Gasca-Lebrón result of 1987 they derive

**Proposition 3.** Let $X = \{x_{\alpha} : |\alpha| = n\} \subset \mathbb{R}^d$ with $\# X = \binom{n+d}{n}$ (i.e. points with different indices are different) and let $Y = \{y_0, \ldots, y_d\} \subset X$ be a set of affinely independent points. For $j = 0, 1, \ldots, d$ let $X_j := X \setminus \langle Y \setminus \{y_j\} \rangle$ where $\langle U \rangle$ denotes the affine hull of $U$. If for $j = 0, \ldots, d$ $X_j$ is a $\mathbb{P}_{n-1}(\mathbb{R}^d)$-correct subset of $X$ then $X$ is
\[ P_n\text{-correct and for all } f \in \mathbb{R}^X \]
\[ p(\cdot, X, P_n(\mathbb{R}^d), f) = \sum_{j=0}^{d} \lambda_j, y(\cdot) \cdot p(\cdot, X_j, P_{n-1}(\mathbb{R}^d), f). \]

From this they derive

**Proposition 4.** Let \( X = \{ x_\alpha : |\alpha| = n \} \subset \mathbb{R}^d \) with \( \#X = \binom{n+d}{n} \). For any \( \alpha \) with \( |\alpha| = n - k \) \( k = 1, \ldots, n \) let \( Y_\alpha := \{ x_{\alpha+k \varepsilon_j} : j = 0, \ldots, d \} \) and
\[ X_\alpha^k := \{ x_{\alpha+\beta} : |\beta| = k \}. \]

(a) If for any \( \alpha \) with \( |\alpha| = n - k \) \( k = 1, \ldots, n \) the set \( Y_\alpha \) is \( P_1(\mathbb{R}^d) \)-correct and\n\[ \langle Y_\alpha \backslash \{ x_{\alpha+\varepsilon_j} \} \rangle = X_\alpha^{k-1} \]
then \( X_\alpha^k \) is \( P_k(\mathbb{R}^d) \)-correct. The recursion
\[ p_\alpha^0(x) := f(x_\alpha), \quad |\alpha| = n, \]
(14) \[ p_\alpha^k(x) := \sum_{j=0}^{d} \lambda_j, y_\alpha(x) \cdot p_{\alpha+\varepsilon_j}^{k-1}(x), \quad |\alpha| = n - k, \quad k = 1, \ldots, n \]
provides successively the interpolants \( p_\alpha^k = p(\cdot, X_\alpha^k, P_k(\mathbb{R}^d), f). \)

(b) If \( X \) is a principal lattice then (13) holds and the recursion (14) ends with the interpolant \( p_0^n = p(\cdot, X, P_n(\mathbb{R}^d), f) \) for any \( f \in \mathbb{R}^X \).

**Definition 5.** A set \( X \subset \mathbb{R}^d \) is called an Aitken-Neville set of degree \( n \) if it can be indexed as \( X = \{ x_\alpha : |\alpha| = n \} \) that for all \( \alpha \) with \( |\alpha| = n - k \) the set \( Y_\alpha := \{ x_{\alpha+k \varepsilon_j} : j = 0, \ldots, d \} \) is \( P_1(\mathbb{R}^d) \)-correct and for all \( \emptyset \neq J \subset \{ 0, \ldots, d \} \)
\[ \gamma \in [\alpha+k \varepsilon_j : j \in J] \implies x_\gamma \in \langle x_{\alpha+k \varepsilon_j} : j \in J \rangle. \]

Any principal lattice is an Aitken-Neville set (but not conversely).

**Definition 6.** A set \( X \subset \mathbb{R}^d \) is called a generalized principal lattice of degree \( n \) if it can be indexed as \( X = \{ x_\alpha : |\alpha| = n \} \) such that for some collection of distinct hyperplanes
\[ H^j_k : k = 0, \ldots, n-1, \quad j = 0, \ldots, d \]
for all \( \alpha, r, i \) with \( |\alpha| = n \)
\[ \alpha_r < n \implies x_\alpha \in H^r_\alpha \quad \text{and} \quad x_\alpha \in H^j_k \implies a_j = k. \]
The concept of generalized principal lattice was introduced by Gasca and Carnicer in [37],[38]. Their definition was then simplified by C. de Boor. Recently [19] Gasca and Carnicer have found this even more simple form.

Each principal lattice is a generalized principal lattice. The hyperplanes are

\[ H^j_k = \{ x \in \mathbb{R}^d : \lambda_j(x) = \frac{k}{n} \} \quad \text{and} \quad X \cap H^j_k = \{ x_\alpha : |\alpha| = n, \ a_j = k \}, \ \text{id est} \ x_\alpha \in H^j_k \iff x_j = k. \]

Carnicer, Gasca and Sauer [18] have shown that every generalized principal lattice is an Aitken-Neville set (but not conversely).

Already in 1977 K. C. Chung and T. H. Yao [35] have introduced a geometric characterization \((GC_n)\) of \(P_n(\mathbb{R}^d)\)-correct sets whose Lagrange polynomials are products of first degree polynomials.

**Definition 7.** A set \(X\) of \(\binom{n+d}{d}\) points of \(\mathbb{R}^d\) is a \(GC_n\) set if for each \(x \in X\) there exists a set of \(n\) hyperplanes such that their union contains \(X\setminus\{x\}\) and not \(x\).

Carnicer, Gasca and Sauer [18] have shown that every Aitken-Neville set of degree \(n\) is a \(GC_n\) set. Another family of \(GC\) sets are natural lattices.

**Definition 8.** A natural lattice of degree \(n\) is a set

\[ X = \{ x_J : J \subset \{1,\ldots,n+d\}, \ \#J = d \} \]

obtained from \(n+d\) hyperplanes \(H_i, \ i = 1,\ldots,n+d\), in general position such that

\[ \{ x_J \} = \cap_{j \in J} H_j, \ \ J \subset \{1,\ldots,n+d\}, \ \#J = d. \]

A natural lattice of degree \(n\) is a \(GC_n\) set hence \(P_n(\mathbb{R}^d)\) correct. The interpolant of a \(GC_n\) set obtains from a Lagrange formula

\[ p(z, X, P_n(\mathbb{R}^d), f) = \sum_{J \subset \{1,\ldots,n+d\}, \#J = d} f(x_J) \prod_{j \notin J} \frac{h_j(z)}{h_j(x_J)} \]

where \(h_j(z) = 0\) is the equation of \(H_j\). Carnicer and Gasca have shown that this interpolant can also be constructed by an Aitken-Neville formula. Choose \(d+1\) hyperplanes among \(H_1,\ldots,H_{n+d}\), for instance take \(J := \{n, n+1,\ldots,n+d\}\). Let \(Y = X_J\) be the simplex whose facets are \(H_j, \ j \in J\). Then \(X_J\) consists of points \(x_{J(j)}\), \(J(j) := J \setminus \{j\}\), having barycentric coordinates \(\lambda_{j,X_J}(z) = \frac{h_j(z)}{h_j(x_{J(j)})}\). The sets \(X \setminus H_j\) are \(P_{n-1}(\mathbb{R}^d)\)-correct, because they are natural lattices of degree \(n-1\). Proposition 2 gives

\[ p(\cdot, X, P_n(\mathbb{R}^d), f) = \sum_{j=n}^{n+d} \frac{h_j(\cdot)}{h_j(x_{J(j)})} \cdot p(\cdot, X \setminus H_j, P_{n-1}(\mathbb{R}^d), f). \]
Definition 9. A hyperplane is called maximal for $X$ if it contains $\binom{n+d-1}{d-1}$ points.

Carnicer and Gasca have got the result

Proposition 10. Let $X$ be a $GC_n$ set in $\mathbb{R}^d$. Assume that $H_0, \ldots, H_d$ are $d+1$ distinct maximal hyperplanes for $X$. Then $X_j := X \setminus H_j$, $j = 0, \ldots, d$, are $GC_{n-1}$ sets and

$$p\left(\cdot, X, \mathbb{P}_n(\mathbb{R}^d), f\right) = \sum_{j=0}^{d} h_j(\cdot) \cdot p\left(\cdot, X_j, \mathbb{P}_{n-1}(\mathbb{R}^d), f\right)$$

where $h_j(z) = 0$ represents $H_j$ and \{y_j\} = \bigcap_{k \neq j} H_k$, $j = 0, \ldots, d$.

In [20] Gasca and Maeztu have made a

Conjecture: Each planar $GC$ set contains a maximal line.

The conjecture till today has been verified only for degrees $n \leq 4$. Carnicer and Gasca [36] have shown that, if it is true, then each planar $GC_n$ set contains at least 3 maximal lines. Therefore, the conjecture together with Proposition 9 implies that any planar $GC_n$ set would allow an Aitken-Neville procedure, too. If the conjecture is true, then for any planar $GC$ set the interpolant could be computed by each of the procedures connected with the names of Lagrange, Newton, Aitken-Neville as in the univariate case. Again, Gasca and Carnicer have shown that this definitely holds for interpolation in $\mathbb{R}^d$ with respect to natural lattices and principal lattices.

Bibliography


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