Efficient divergence-conforming finite element methods for incompressible flows

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Abstract

In this talk we consider the discretization of the unsteady incompressible Navier-Stokes equations in a velocity-pressure formulation:

$$\begin{cases} \frac{\partial}{\partial t}u + \operatorname{div}(-\nu\nabla u + u\otimes u + pI) & = f \quad \text{in } \Omega \\ \operatorname{div} u & = 0 \quad \text{in } \Omega \end{cases} \tag{1}$$

with boundary conditions $u = u_D$ on $\Gamma_D \subset \partial\Omega$ and $(\nu\nabla u - pI)\cdot n = 0$ on $\Gamma_{out} = \partial\Omega \setminus \Gamma_D$. Here, $\nu = const$ is the kinematic viscosity, u the velocity, p the pressure, and f is an external body force. We present an efficient, high order accurate and *robust* discretization method based on the following main ingredients:

First, we make a disctinction between stiff linear parts and less stiff non-linear parts with respect to their *temporal and spatial* treatment. We exploit this using *operator-splitting time integration schemes* which rely only on efficient solution strategies for two simpler sub-problems: a corresponding hyperbolic transport problem and an unsteady Stokes problem.

Secondly, for the hyperbolic transport problem a spatial discretization with an Upwind Discontinuous Galerkin (DG) method and an explicit treatment in the time integration scheme is rather natural and allows for an efficient implementation.

Third and most importantly, the discretization of the Stokes problems is tailored with respect to two important challenges: a *compatible* treatment of velocity and pressure in view of the incompressibility constraint and an efficient solution of arising linear systems. In order to fulfill the incompressibility constraint exactly we use an $H(\operatorname{div})$-conforming discretization of the velocity combined with discontinuous pressures. We discuss the two main advantages of such a choice: *energy-stability* and *pressure-robustness*. The enforcement of the tangential continuity of the velocity (weakly) requires a discontinuous Galerkin method which renders the solution of linear systems computationally expensive. To counteract these costs and to obtain an overall efficient scheme we apply several techniques, e.g. hybridization, reduced $H(\operatorname{div})$-conforming finite element spaces and domain decomposition preconditioning.

We conclude the talk with the discussion of the performance on applications from 2D and 3D, laminar and turbulent incompressible flows.