Travelling Waves in Dilatant Non-Newtonian Thin Films

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Abstract. We prove the existence of a travelling wave solution for a gravity-driven thin film of a viscous and incompressible dilatant fluid coated with an insoluble surfactant. The governing system of second order partial differential equations for the film's height h and the surfactant's concentration γ are derived by means of lubrication theory applied to the non-Newtonian Navier–Stokes equations.

1 Introduction

Non-Newtonian fluids are ubiquitious in Nature and technical applications. In contrast to Newtonian fluids non-Newtonian fluids exhibit a shear rate dependent viscosity. A commonly used classification for non-Newtonian viscosities is to distinguish between shear-thinning, shear-thickening and generalised Newtonian fluids. While shear-thinning fluids – whose viscosity decreases with increasing shear stress – appear as natural as well as industrial liquids, shear-thickenning fluids are mostly observed as suspensions. In this work we treat only shear-thickening or *dilatant* fluids, i.e. those whose viscosity increases with increasing shear stress. We consider a thin film of a viscous, incompressible dilatant fluid on a horizontal impermeable bottom, carrying a layer of insoluble *surface active agents*, also named *surfactant* on its surface. As a consequence of the *Marangoni effect*, saying that the surfactant spread from places with lower surface tension to places with higher surface tension, the presence oft surfactant leads to a diminution of the surface tension. In mathematical parlance, the surface tension is a decreasing function of the surfactant's concentration. Starting from the Navier–Stokes equation we use lubrication theory in oder to describe the hydrodynamic behaviour of the thin film. More precisely, we denote by $\gamma(t,x)$ and $h(t,x)$ the surfactant's concentration and the film's height at time $t \geq 0$ and position $x \in \mathbb{R}$, respectively, by G the gravitational force, and by d_0 a positive diffusion coefficient. Furthermore, we specify the viscosity function to be a first-order approximation

$$\mu(d) = \mu_0 + \mu_1 d, \quad d \geq 0,$$

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of a general dilatant non-Newtonian fluid, where μ_0 and μ_1 are positive[1]. Finally, introducing the surface tension $\sigma = \sigma(\gamma(t,x))$ we may formulate the evolution equations

$$h_t + \frac{d}{dx}\left(\frac{1}{2\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x h^2 - \frac{G}{3\mu_0}h_x h^3\right) = 0, \quad t>0,\ x\in\mathbb{R}, \tag{1}$$

$$\gamma_t + \frac{d}{dx}\left(\frac{1}{\mu_0}h\gamma\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{2\mu_0}h^2 h_x\gamma - d_0\gamma_x\right) = 0, \quad t>0,\ x\in\mathbb{R}, \tag{2}$$

for the film's height h and the surfactant's concentration γ. Note that we deal with a gravity driven system of second order, neglecting fourth-order capillary effects.
To the best of our knowledge there are no rigorous analytic results available about non-Newtonian thin-film equations including the presence of surfactant. For results on non-Newtonian thin films without the influence of surfactant we refer the reader to the works [12, 13], where the spreading of general power-law fluids and travelling wave solutions for thin films of power-law fluids are considered, respectively. In [2] the authors study self-similar solutions in free-boundary problems for non-Newtonian power-law fluids. Moreover, in [15] the authors use lubrication approximation to derive the governing equations of a power law non-Newtonian liquid flowing on an inclined plane and they study qualitative properties of the corresponding travelling waves solutions.
For the Newtonian case, we mention the paper [8], where the authors study a coupled system of parabolic equations, consisting of a degenerate fourth-order equation for the film's height and a second-order equation for the surfactant's concentration. For further results on the Newtonian case see [3, 6, 9, 11, 14, 16] and the references therein. Finally, we refer to the contribution [7], in which the authors prove the existence of travelling waves in a Newtonian thin film with surfactant. In our work we verify that the results of the latter case are even true in the non-Newtonian case of a shear-thickening fluid. More precisely, we prove the following result:

Theorem
There exists a C^2-travelling wave solution to (1)–(2) *connecting a fully coated state, ($\gamma \sim 1$) with an uncoated state ($\gamma \sim 0$). More precisely, there is a pair of functions* $(h_0, \gamma_0) \in C^2(\mathbb{R}, \mathbb{R}^2)$ *such that*

$$\lim_{\xi\to-\infty}\gamma_0(\xi) = 1, \qquad \lim_{\xi\to\infty}\gamma_0(\xi) = 0$$

and such that $h(t,x) := h_0(x-ct), \gamma(t,x) := \gamma_0(x-ct)$ satisfies (1)–(2) *pointwise. In addition, γ_0 is monotonically decreasing and h_0 is positive, bounded, and bounded away from zero.*

[1] A further smallness condition on the ratio μ_1/μ_0 is need in our analysis.

2 Modelling with a First-Order Approximation of the Viscosity

We consider the dynamics of a thin liquid film of a viscous incompressible non-Newtonian fluid being located on a horizontal plane. The film's free upper surface is coated with an insoluble surfactant. The presence of surfactants leads to a lowering of the surface tension which induces surface gradients and Marangoni stresses. Both effects have a strong impact on the dynamics of the thin film.

Before introducing the governing equations we mention that in our investigation all ingredients are assumed to be homogeneous in one lateral direction so that we may in fact restrict our analysis to a two-dimensional cross section of the fluid layer.

Denoting by x and z the horizontal and vertical space variable we consider the horizontal impermeable bottom to be located at height $z = 0$ and the upper free surface of the liquid film at $z = h(t, x)$, respectively. Let furthermore $v(t, x, z) = (u(t, x, z), w(t, x, z))$ denote the velocity field, $p(t, x, z)$ the pressure of the fluid, respectively. As we are dealing with an incompressible fluid we assume the fluid's density $\rho > 0$ to be constant. Moreover,

$$D = \frac{1}{2}\begin{pmatrix} 2u_x & u_z + w_x \\ u_z + w_x & 2w_z \end{pmatrix}$$

is the so-called rate of strain tensor and the trace of D^2 is given by

$$|D|^2 = u_x^2 + w_z^2 + \frac{1}{2}(u_z + w_x)^2.$$

To give consideration to the fact that the fluid is assumed to be non-Newtonian, we introduce the viscosity function μ. Finally, $g = (0, G)$ is the gravitational force. The dynamics of the viscous, incompressible non-Newtonian fluid is thus prescribed by the *Navier–Stokes equation*

$$\rho\big(v_t + (v \cdot \nabla)v\big) = \operatorname{div}\big(\mu(|D|^2)D\big) - \nabla p - \rho g \quad \text{in } \Omega, \tag{3}$$

together with the *continuity equation*

$$u_x + w_z = 0 \quad \text{in } \Omega, \tag{4}$$

where Ω is the region bounded by the impermeable bottom at $z = 0$ and the free surface at $z = h(t, x)$. In addition, we impose the *no-slip boundary condition* on the impermeable bottom, i.e. we have

$$u = w = 0 \quad \text{on } z = 0, \tag{5}$$

whereas we prescribe the *kinematic boundary condition*

$$h_t + uh_x = w \quad \text{on } z = h(t, x) \tag{6}$$

at the free surface. As a consequence of the balance of stresses we obtain the boundary condition

$$Tn = \sigma\kappa n + \nabla_s\sigma \quad \text{on } z = h(t, x), \tag{7}$$

where $T = \mu\big(\sqrt{2|D|^2}\big)D - p$ denotes the stress tensor of the fluid, κ the free surface's curvature, n the outer pointing normal and σ the γ-dependent surface tension, respectively. Moreover, $\nabla_s \sigma$ is the surface gradient. Lastly, the surfactant spreads on the free surface at $z = h(t,x)$ according to the *advection transport equation*

$$\gamma_t + \big(u\gamma - d_0\gamma_x\big)_x = 0 \quad \text{on } z = h(t,x), \tag{8}$$

with $d_0 > 0$ being the surface diffusion coefficient.
Defining H and L as a characteristic height and length of the film, respectively, we use lubrication theory in order to simplify the above system of equations. More precisely, based on the assumption that the film's length L is fairly large compared to the film's height H in the undisturbed setting, by an expansion in $\varepsilon = H/L$ we obtain the leading order system[2] (9)–(16) emanating from (3)–(8).

$$\begin{aligned}
-p_x + (\mu(|u_z|)u_z)_z &= 0 && \text{in } \Omega && (9)\\
-p_z - G &= 0 && \text{in } \Omega && (10)\\
u_x + w_z &= 0 && \text{in } \Omega && (11)\\
u = w &= 0 && \text{on } z = 0 && (12)\\
p &= 0 && \text{on } z = h(t,x) && (13)\\
u_z &= \sigma(\gamma)_x && \text{on } z = h(t,x) && (14)\\
h_t + uh_x &= w && \text{on } z = h(t,x), && (15)\\
\gamma_t + \big(u\gamma - d_0\gamma_x\big)_x &= 0 && \text{on } z = h(t,x) && (16)
\end{aligned}$$

Here (9)–(10) emanate from the *Navier–Stokes equation*. Moreover, (13) originates from the normal stress balance in (7), together with the assumption[3] that capillary effects of fourth order are negelcted, while (14) comes from the tangential stress balance condition in (7).

In the sequel we solve the system obtained by lubrication approximation for the pressure p and the velocity field $(u(t,x,z), w(t,x,z))$ so that we finally end up with two evolution equations – one for the film's height $h(t,x)$ and one for the surfactant's concentration $\gamma(t,x)$.

Determination of the pressure p.
We start by computing the pressure p by exploiting (9)–(16). To this end, an integration of (10) from z to $h(t,x)$ yields

$$p(t,x,h(t,x)) - p(t,x,z) = G\big(z - h(t,x)\big).$$

[2]Note that this system is in dimensionless form but for the sake of better readability we omit the introduction of new notation.

[3]For more details on the modelling of the spreading of the surfactant see [10].

Using the boundary condition (13), saying that $p\big(t,x,h(t,x)\big) = 0$ we obtain[4]

$$p(t,x,z) = G\big(h(t,x) - z\big) \quad \text{in } \Omega, \tag{17}$$

i.e. we have an expression for p in terms of the height h and the variable z only.

Determination of the horizontal velocity u.
It remains to solve the system for the velocity field $(u(t,x,z), w(t,x,z))$. To this end, we take the derivative of (17) with respect to x so that

$$p_x(t,x,z) = Gh_x(t,x) \quad \text{in } \Omega$$

and insert the result into (9). This leads to the identity

$$0 = -p_x + (\mu(|u_z|)u_z)_z = -Gh_x(t,x) + (\mu(|u_z|)u_z)_z.$$

Another integration of this equation from z to $h(t,x)$ yields

$$\mu\big(|u_z(t,x,h(t,x))|\big)u_z(t,x,h(t,x)) - \mu\big(|u_z(t,x,z)|\big)u_z(t,x,z) = Gh_x(t,x)\big(h(t,x) - z\big)$$

Using the boundary condition (14) this equation becomes

$$\mu\big(|u_z(t,x,z)|\big)u_z(t,x,z) = \mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - Gh_x(t,x)\big(h(t,x) - z\big). \tag{18}$$

From now on we approximate the general viscosity function μ by

$$\mu(d) = \mu_0 + \mu_1 d, \quad d \geq 0,$$

where $\mu_0, \mu_1 > 0$ are positive. Inserting this function into (18), we obtain

$$\left(1 + \frac{\mu_1}{\mu_0}|u_z|\right) u_z = \frac{1}{\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{\mu_0}h_x(t,x)\big(h(t,x) - z\big). \tag{19}$$

Denoting the right-hand side of (19) by

$$A := \frac{1}{\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{\mu_0}h_x(t,x)\big(h(t,x) - z\big)$$

we arrive at

$$u_z(t,x,z) = \operatorname{sgn}(A)\frac{\mu_0}{2\mu_1}\left|1 - \sqrt{1 + 4\frac{\mu_1}{\mu_0}|A|}\right|.$$

Claiming that $\mu_1/\mu_0 \ll 1$ is rather small we use a first-order Taylor approximation of $\sqrt{1+x}$ around $x = 0$ to obtain

$$\begin{aligned} u_z &= \operatorname{sgn}(A)\frac{\mu_0}{2\mu_1}\left|1 - \sqrt{1 + 4\frac{\mu_1}{\mu_0}|A|}\right| \\ &\approx \operatorname{sgn}(A)\frac{\mu_0}{2\mu_1}\left|1 - 1 - 2\frac{\mu_1}{\mu_0}|A|\right| \\ &= \operatorname{sgn}(A)|A| \\ &= A. \end{aligned}$$

[4]Note that this tells us in particular that $p(t,x,z) > 0$ for all $z < h(t,x)$.

That is, we have

$$u_z(t,x,z) = \frac{1}{\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{\mu_0}h_x(h-z).$$

Integration from 0 to z and exploiting the no-slip boundary condition for u then leads to the equation

$$u(x,z) = z\frac{1}{\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{\mu_0}h_x\left(hz - \frac{z^2}{2}\right). \tag{20}$$

Determination of the evolution equations for γ and h.
Evaluating $u(x,z)$ in (20) at $z = h(t,x)$ yields

$$u(x,h) = h\frac{1}{\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{2\mu_0}h^2h_x.$$

Inserting this expression for $u(x,h)$ into (16) gives us the evolution equation for the surfactant's concentration γ, namely

$$\gamma_t + \frac{d}{dx}\left(\frac{1}{\mu_0}h\gamma\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{2\mu_0}h^2h_x\gamma - d_0\gamma_x\right) = 0, \quad t>0,\ x\in\mathbb{R}.$$

To obtain the evolution equation for the film's height h we integrate (20) with respect to z from 0 to $h(t,x)$. This yields

$$\int_0^h u(x,z)\,dz = \frac{1}{2\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x h^2 - \frac{G}{3\mu_0}h_x h^3$$

and finally the evolution equation (15) for the film's height h becomes

$$h_t + \frac{d}{dx}\left(\frac{1}{2\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x h^2 - \frac{G}{3\mu_0}h_x h^3\right) = 0, \quad t>0,\ x\in\mathbb{R}.$$

Summarising all, we end up with the following system of equations

$$h_t + \frac{d}{dx}\left(\frac{1}{2\mu_0}\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x h^2 - \frac{G}{3\mu_0}h_x h^3\right) = 0, \quad t>0,\ x\in\mathbb{R}, \tag{21}$$

$$\gamma_t + \frac{d}{dx}\left(\frac{1}{\mu_0}h\gamma\mu\big(|\sigma(\gamma)_x|\big)\sigma(\gamma)_x - \frac{G}{2\mu_0}h^2h_x\gamma - d_0\gamma_x\right) = 0, \quad t>0,\ x\in\mathbb{R}. \tag{22}$$

2.1 Remark

Let us briefly comment on the choice $\mu(d) = \mu_0 + \mu_1 d$, $d \geq 0$, for the viscosity function. We use this as an approximation of the viscosity of a general shear-thickening fluid, as $\mu_1 > 0$. Note that the positivity of μ_1 is in particular used in Section 3, where we analyse the dynamical system induced by a travelling wave ansatz in (21)–(22).

3 Existence of Travelling Waves

In this section we investigate the existence of travelling wave solutions for (21)–(22), i.e. solutions of the form

$$(h,\gamma)(t,x) = (H,\Gamma)(x-ct), \quad t \in (0,\infty),\ x \in \mathbb{R},$$

where $c > 0$ denotes the positive wave speed. With this ansatz and introducing the new variable $\xi := x - ct$ the equations (21)–(22) become

$$\begin{aligned} -cH' + \frac{d}{dx}\left(\frac{1}{2}H^2\sigma'\Gamma'\left(1+\frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) - \frac{G}{3\mu_0}H^3H'\right) &= 0, \quad \xi \in \mathbb{R}, \\ -c\Gamma' + \frac{d}{dx}\left(H\Gamma\sigma'\Gamma'\left(1+\frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) - \frac{G}{2\mu_0}H^2H'\Gamma - d_0\Gamma'\right) &= 0, \quad \xi \in \mathbb{R}, \end{aligned}$$

and integration with respect to ξ over $\mathbb{R}$ yields

$$cH - \frac{1}{2}H^2\sigma'\Gamma'\left(1+\frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) + \frac{G}{3\mu_0}H^3H' = \kappa_1, \quad \xi \in \mathbb{R}, \tag{23}$$

$$c\Gamma - H\Gamma\sigma'\Gamma'\left(1+\frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) + \frac{G}{2\mu_0}H^2H'\Gamma + d_0\Gamma' = \kappa_2, \quad \xi \in \mathbb{R}, \tag{24}$$

with constants $\kappa_1, \kappa_2 \in \mathbb{R}$.

3.1 Remark
Note that (23)–(24) is a system of ordinary differential equations only if $G > 0$. More precisely, if $G = 0$, i.e. if gravity is absent, then all derivatives H' of H vanish.

3.1 The system of explicit ordinary differential equations

We are interested in finding bounded travelling waves that connect a fully coated state ($\Gamma \sim 1$) with an uncoated state ($\Gamma \sim 0$). More precisely, we expect H to be nonnegative with $\sup_{\xi\in\mathbb{R}} H < \infty$, Γ to decay to zero as $\xi \to \infty$ and $\Gamma \to 1$ as $\xi \to -\infty$ with $0 < \Gamma(\xi) < 1$ for $\xi \in \mathbb{R}$. Furthermore, we assume the surface tension σ to be a decaying function of the concentration Γ with

$$\sigma \in C^3\left([0,1)\right), \quad \sigma' < 0 \text{ on } [0,1) \quad \text{and} \quad \lim_{\Gamma\to 1} \sigma' \to -\infty. \tag{25}$$

Additionally assuming that

$$\limsup_{\xi\to\infty} H' < \infty \quad \text{and} \quad \lim_{\xi\to\infty} \Gamma' = 0$$

one may conclude that $\kappa_2 = 0$ in (24). In view of the previously stated properties we seek for global (i.e. for all $\xi \in \mathbb{R}$ existing) bounded solutions to the system

$$cH - \frac{1}{2}H^2\sigma'\Gamma'\left(1 + \frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) + \frac{G}{3\mu_0}H^3H' = \kappa_1, \quad \xi \in \mathbb{R}, \tag{26}$$

$$c\Gamma - H\Gamma\sigma'\Gamma'\left(1 + \frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) + \frac{G}{2\mu_0}H^2H'\Gamma + d_0\Gamma' = 0, \quad \xi \in \mathbb{R}, \tag{27}$$

$$H \geq 0, \quad 0 < \Gamma < 1, \quad \xi \in \mathbb{R}, \tag{28}$$

satisfying the additional properties

$$\lim_{\xi \to -\infty} \Gamma(\xi) = 1 \qquad \text{and} \qquad \lim_{\xi \to \infty} \Gamma(\xi) = 0.$$

Multiplying (26) by -2Γ and (27) by H and adding the resulting equations, we obtain the identity

$$d_0H\Gamma' - \frac{G}{6\mu_0}H^3\Gamma H' + \Gamma(2\kappa_1 - cH) = 0.$$

Solving this equation for H' gives us

$$H' = \frac{6\mu_0}{G}\frac{d_0H\Gamma' + \Gamma(2\kappa_1 - cH)}{H^3\Gamma}. \tag{29}$$

Moreover, multiplying (26) by 6Γ and (27) by $-4H$ and adding the resulting equations, we have

$$2\Gamma(cH - 3\kappa_1) + \sigma'\Gamma'H^2\Gamma\left(1 + \frac{\mu_1}{\mu_0}|\sigma'\Gamma'|\right) - 4d_0H\Gamma' = 0,$$

which may be rewritten in the equivalent form

$$A_1(H,\Gamma)|\Gamma'|\Gamma' + A_2(H,\Gamma)\Gamma' + A_3(H,\Gamma) = 0, \tag{30}$$

where

$$A_1 := \frac{\mu_1}{\mu_0}H^2(\sigma')^2\Gamma, \quad A_2 := H(4d_0 - H\Gamma\sigma') \quad \text{and} \quad A_3 := 2\Gamma(3\kappa_1 - cH).$$

In order to solve (30) we provide the following auxiliary results. Note that in the proof of Lemma 3.2 it turns out that κ_1 has to be positive in order to guarantee the positivity of H.

3.2 Lemma

Assume that (H,Γ) is a global bounded C^1-solution of (26) – (28). Then given $\xi_* \in \mathbb{R}$ such that $\Gamma'(\xi_*) < 0$, we have that $\Gamma'(\xi) < 0$ for all $\xi \geq \xi_*$.

Proof. Let $\xi_* \in \mathbb{R}$ such that $\Gamma'(\xi_*) < 0$. Assume by contradiction that there is a $\xi^* > \xi_*$ such that $\Gamma'(\xi^*) \geq 0$. One may choose ξ^* minimal in the sense that $\Gamma'(\xi) < 0$ for all $\xi \in [\xi_*, \xi^*)$ and $\Gamma'(\xi^*) = 0$. Invoking (30) and the positivity of A_1 and A_2 one knows that $H(\xi) < 3\kappa_1/c$ for all $\xi \in [\xi_*, \xi^*)$ and $H(\xi^*) = 3\kappa_1/c$. This implies that $H'(\xi^*) \geq 0$. On the other hand (27) also shows that $H'(\xi^*) < 0$, which is impossible. □

3.3 Proposition
Assume that (H, Γ) is a global bounded C^1-solution of (26) – (28). *Then $\Gamma(\xi)$ is strictly decreasing on $\mathbb{R}$.*

Proof. Let $N := \{\xi \in \mathbb{R}\,;\, \Gamma'(\eta) < 0 \text{ for all } \eta > \xi\}$ and define $\xi_* := \inf N$. As Γ is decreasing from 1 to 0 there doubtlessly exists $\xi^* \in \mathbb{R}$ with $\Gamma'(\xi^*) < 0$. Thanks to Lemma 3.2 we know that $\xi^* \in N$ so that in particular $N \neq \emptyset$ and $\xi_* \leq \xi^*$. We aim to show that $\xi_* = -\infty$. Assume by contradiction that $\xi_* > -\infty$. Then Lemma 3.2 and the definition of ξ_* imply that $\Gamma'(\xi) \geq 0$ for all $\xi \leq \xi_*$ which is impossible since $\lim_{\xi \to -\infty} \Gamma(\xi) = 1$ and $\Gamma(\xi_*) < 1$. □

The above reasoning implies that any global bounded C^1-solution (H, Γ) to (26)–(28) satisfies $\Gamma' < 0$ and the height H obeys the bound

$$H < \frac{3\kappa_1}{c}. \tag{31}$$

Under the assumption (31) the above equation (30) may be uniquely resolved for $\Gamma' = \big(A_2 - \sqrt{A_2^2 + 4A_1A_3}\big)/2A_1$, i.e.

$$\Gamma' = -\frac{\mu_0(4d_0 - H\Gamma\sigma')}{2\mu_1 H\Gamma(\sigma')^2}\left(\sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{\Gamma^2(\sigma')^2(3\kappa_1 - cH)}{(4d_0 - H\Gamma\sigma')^2}} - 1\right).$$

Inserting the expression for Γ' into the equation (29) and introducing for the sake of simplicity the notation

$$\rho(z) := -\frac{1}{\sigma'(z)}, \quad z \in [0,1), \tag{32}$$

we end up with the system

$$H' = -\frac{6\mu_0}{GH^3}\left[\frac{d_0\mu_0(4d_0\rho + H\Gamma)\rho}{2\mu_1\Gamma^2}\left(\sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{\Gamma^2(3\kappa_1 - cH)}{(4d_0\rho + H\Gamma)^2}} - 1\right) - (2\kappa_1 - cH)\right] \tag{33}$$

$$\Gamma' = -\frac{\mu_0(4d_0\rho + H\Gamma)\rho}{2\mu_1 H\Gamma}\left(\sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{\Gamma^2(3\kappa_1 - cH)}{(4d_0\rho + H\Gamma)^2}} - 1\right) \tag{34}$$

of differential equations.
In order to construct a travelling wave solution for (21)–(22) we now introduce the vector field $(f_1, f_2)(H, \Gamma)$ according to (33)–(34). More precisely, using from now on the notation ρ for a C^2-extension to $[0, \infty)$ of ρ, defined in (32), we introduce the function

$$g(H, \Gamma) := \frac{\mu_0(4d_0\rho + H\Gamma)\rho}{2\mu_1\Gamma}\left(\sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{\Gamma^2(3\kappa_1 - cH)}{(4d_0\rho + H\Gamma)^2}} - 1\right) \tag{35}$$

for $(H, \Gamma) \in (0, \infty) \times (0, \infty)$ and set $f := (f_1, f_2)$ with

$$f_1(H,\Gamma) := -\frac{6\mu_0}{GH^3}\left(\frac{d_0}{\Gamma}g(H,\Gamma) - (2\kappa_1 - cH)\right), \tag{36}$$

$$f_2(H,\Gamma) := -\frac{1}{H}g(H,\Gamma). \tag{37}$$

Note that by the regularity of σ or ρ, respectively, this vector field clearly enjoys the regularity

$$(f_1, f_2) \in C^2\big((0,\infty)\times(0,\infty), \mathbb{R}^2\big),$$

such that given any $(H_0,\Gamma_0) \in (0,\infty)\times(0,\infty)$, the *Theorem of Picard–Lindelöf* provides the existence of a unique nonextendable positive solution

$$(H,\Gamma) \in C^2\big((\xi_-,\xi_+), \mathbb{R}^2\big)$$

to (33)–(34), together with the initial conditions

$$H(0) = H_0 \qquad \text{and} \qquad \Gamma(0) = \Gamma_0. \tag{38}$$

3.4 Remark

- *Sending μ_1 to zero in* (33)–(34) *by applying the Theorem of de l'Hôspital we obtain the same system of equations as for the Newtonian case in [7, Eqs. (4.15a)–(4.15b)], where the authors have set $\mu_0 = 1$. More precisely, we have*

$$\lim_{\mu_1\to 0} f_1(H,\Gamma) = \frac{6\mu_0 H\Gamma(2\kappa_1 - cH) - 12\mu_0 d_0(cH-\kappa_1)\rho}{GH^3(H\Gamma + 4d_0\rho)}$$

and

$$\lim_{\mu_1\to 0} f_2(H,\Gamma) = \frac{2\Gamma(cH - 3\kappa_1)\rho}{H(H\Gamma + 4d_0\rho)}.$$

- *There are widely accepted models for the surface tension σ which possess the properties stated in* (25)*. We mention the laws*

$$\sigma(\gamma) := \sigma_0 + \sigma_1\ln(1-\gamma), \quad \sigma(\gamma) := \sigma_0 + \sigma_1\ln(1-\gamma) + \sigma_2\gamma^2, \tag{39}$$

of Szyszkowski and Frumkin, respectively, where $\sigma_0,\sigma_1,\sigma_2 > 0$ are positive constants (see i.e. [4, 5]). If σ is assumed to follow the Szyszkowski law, then (25) *is satisfied. Moreover, if σ behaves as prescribed by Frumkin's law, then* (25) *is satisfied for all positive constants $\sigma_0,\sigma_1,\sigma_2 > 0$ with $2\sigma_1 > \sigma_2$. Finally we mention that for the common choise $\sigma = \sigma_0(1-\gamma)$ the last condition in* (25) *is violated.*

3.2 Critical points of the system

In this section we study the steady states of the above derived system of ordinary differential equations. On the one hand we shall see that the vector field (f_1, f_2) possesses a null in the interior of the phase space. On the other hand the vector field vanishes in one point of the boundary. In both cases we refer to these points as critical points of the system:

3.5 Definition (Critical points)
We call

$$y_\alpha := (2\kappa_1/c, 1) \quad \textit{and} \quad y_\omega := (\kappa_1/c, 0)$$

critical points *of the system* (33)–(34).

This definition is justified by the following two lemmas.

3.6 Lemma
Given $H \in (0, \infty)$, the vector field (f_1, f_2) satisfies

(i) $f_1(H, 1) = -\frac{6\mu_0}{GH^3}(cH - 2\kappa_1)$;

(ii) $f_2(H, 1) = 0$.

Proof. Recalling the definition of ρ these statements immediately follow since $\rho(1) = 0$. □

3.7 Lemma
Given $H \in (0, \infty)$, we have the following limits:

(i) $\lim_{\Gamma \to 0} f_1(H, \Gamma) = -\frac{3\mu_0}{GH^3}(cH - \kappa_1)$;

(ii) $\lim_{\Gamma \to 0} f_2(H, \Gamma) = 0$.

Proof. (i) In view of the definition of f_1 we first calculate $\lim_{\Gamma \to 0} g(H, \Gamma)/\Gamma$. Twice applying the Theorem of de l'Hôspital yields

$$\lim_{\Gamma \to 0} \frac{g(H, \Gamma)}{\Gamma} = \frac{1}{2d_0}(3\kappa_1 - cH).$$

This finally implies

$$\lim_{\Gamma \to 0} f_1(H, \Gamma) = -\frac{6\mu_0}{GH^3}\left(\frac{1}{2}(3\kappa_1 - cH) - (2\kappa_1 - cH)\right) = -\frac{3\mu_0}{GH^3}(cH - \kappa_1).$$

(ii) This may also be obtained by applying the Theorem of de l'Hôspital. □

As a direct consequence of the above two lemmas we obtain the following.

3.8 Corollary (Critical points)
We have

(i) $f(2\kappa_1/c, 1) = (0, 0)$;

(ii) $\lim_{\Gamma\to 0} f(\kappa_1/c, \Gamma) = (0, 0)$.

3.3 Study of the ω-limit set $\omega(H, \Gamma)$

Our aim is to construct a travelling wave solution for (21)–(22) by verifying the existence of a heteroclinic orbit for (33)–(34) connecting the critical points y_α and y_ω. To this end, we show in this section that the maximal positive exit time ξ_+ is infinite and that the system's ω-limit set $\omega(H, \Gamma)$ consists only of the critical point y_ω for all $y = (H, \Gamma)$ in the rectangle

$$R := (\kappa_1/c, 2\kappa_1/c) \times (0, 1).$$

We start by proving the following auxiliary result.

3.9 Lemma
Given $H \in (0, 3\kappa_1/c)$ and $\Gamma > 0$ the following inequality holds true:

$$f_1(H, \Gamma) > -\frac{3\mu_0}{GH^3}(cH - \kappa_1).$$

Proof. Applying the inequality $\sqrt{1+y} - 1 \leq y/2$ for $y \geq 0$ to g, we obtain

$$g(H, \Gamma) \leq 2\Gamma\rho\frac{3\kappa_1 - cH}{4d_0\rho + H\Gamma} < \Gamma\frac{3\kappa_1 - cH}{2d_0}.$$

Inserting this into f_1 leads to the estimate

$$f_1(H, \Gamma) > -\frac{6\mu_0}{GH^3}\left(\frac{1}{2}(3\kappa_1 - cH) - 2\kappa_1 + cH\right) = -\frac{3\mu_0}{GH^3}(cH - \kappa_1)$$

for all $H \in (0, 3\kappa_1/c)$ and $\Gamma > 0$. □

The positive invariance of R is part of the next result.

3.10 Lemma (Positive invariance of R)
Given $(H_0, \Gamma_0) \in R$, let

$$(H, \Gamma) \in C^2\big((\xi_-, \xi_+), \mathbb{R}^2\big)$$

be the nonextendable solution to (33)–(34)*, together with the initial conditions* (38)*. Then the rectangle R is positively invariant, i.e. we have*

$$\big(H(\xi), \Gamma(\xi)\big) \in R$$

for all $\xi \in [0, \xi_+)$.

Proof. Let $(H_0, \Gamma_0) \in R$. We firstly prove that

$$f_1(\kappa_1/c, \Gamma) > 0 \quad \text{and} \quad f_1(2\kappa_1/c, \Gamma) < 0$$

for all $\Gamma \in (0,1)$. Thanks to the fact that $g(2\kappa_1/c, \Gamma)$ is strictly positive for all $\Gamma > 0$ we have

$$f_1(2\kappa_1/c, \Gamma) = -\frac{6\mu_0 d_0}{GH^3\Gamma} g(2\kappa_1/c, \Gamma) < 0, \quad \Gamma > 0.$$

On the other hand, we may use Lemma 3.9 to obtain

$$f_1(\kappa_1/c, \Gamma) > 0$$

for all $\Gamma > 0$. Together with Lemma 3.6 (ii) and Lemma 3.7 (ii) this proves the assertion. $\square$

3.11 Lemma
Let $\alpha \in C\big([0,1], \mathbb{R}\big)$ be given with $\alpha(z) > 0$ for all $z \in [0,1]$. Then there exists $z_* \in (0,1)$ such that

$$\frac{\sqrt{1+\alpha(z)z^2} - 1}{z^2} \geq \frac{\alpha(z)}{2} - z, \quad z \in (0, z_*].$$

Proof. We define $\alpha_* := \min \alpha(z)$ and $\alpha^* := \max \alpha(z)$. Thanks to the positivity of α we may choose

$$0 < z_* < \min\{\frac{\alpha_*}{2}, \frac{8}{\alpha^{*2}}, 1\}.$$

Moreover, given any $z \in (0, z_*]$, we have

$$\frac{\alpha(z)}{2} - z \geq 0 \quad \text{and} \quad 2 \geq z\frac{\alpha^{*2}}{4}.$$

Using these relations in combination with the positivity of α and the definition of α^* we find

$$2 \geq z\frac{\alpha^{*2}}{4} \geq z\frac{\alpha(z)^2}{4} \geq z\left(\frac{\alpha(z)}{2} - z\right)^2$$

for any $z \in (0, z_*]$. From this we conclude that

$$2z^3 \geq z^4\left(\frac{\alpha(z)}{2} - z\right)^2 = \left(\frac{\alpha(z)}{2}z^2 - z^3\right)^2, \quad z \in (0, z_*],$$

which in turn implies

$$1 + \alpha(z)z^2 \geq 1 + \alpha(z)z^2 - 2z^3 + \left(\frac{\alpha(z)}{2}z^2 - z^3\right)^2 = \left(1 + \frac{\alpha(z)}{2}z^2 - z^3\right)^2, \quad z \in (0, z_*],$$

and finally

$$\sqrt{1+\alpha(z)z^2} \geq 1 + \frac{\alpha(z)}{2}z^2 - z^3, \quad z \in (0, z_*],$$

which is equivalent to

$$\frac{\sqrt{1+\alpha(z)z^2}-1}{z^2} \geq \frac{\alpha(z)}{2} - z, \quad z \in (0, z_*].$$

□

We now set

$$\alpha(\Gamma) := 8\frac{\mu_1}{\mu_0}\frac{3\kappa_1 - cH}{(4d_0\rho(\Gamma)+H\Gamma)^2}, \quad \Gamma \in (0,1), H \in (0, 3\kappa_1/c),$$

and choose $\Gamma_* \in (0,1)$ according to Lemma 3.11. Moreover, defining

$$\rho_* := \min_{z\in[0,\Gamma_*]} \rho(z) \quad \text{and} \quad \rho^* := \max_{z\in[0,1]} \rho(z)$$

we have that $\rho_* > 0$ and $\rho^* < \infty$.

3.12 Lemma
There exist $\Gamma_* \in (0,1)$ and a constant $C > 0$, independent of H and Γ, such that

$$f_1(H,\Gamma) + \frac{3\mu_0}{GH^3}(cH - \kappa_1) \leq C\Gamma$$

for all $(H,\Gamma) \in (\kappa_1/c, 2\kappa_1/c) \times (0, \Gamma_*]$.

Proof. Based on Lemma 3.11 we may derive the estimate

$$\begin{aligned}
&\frac{\mu_0 d_0(4d_0\rho + H\Gamma)\rho}{2\mu_1\Gamma^2}\left(1 - \sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{3\kappa_1 - cH}{(4d_0\rho+H\Gamma)^2}\Gamma^2}\right) + \frac{1}{2}(3\kappa_1 - cH)\\
&\leq \frac{\mu_0 d_0(4d_0\rho + H\Gamma)\rho}{2\mu_1}\left(4\frac{\mu_1}{\mu_0}\frac{cH - 3\kappa_1}{(4d_0\rho+H\Gamma)^2} + \Gamma\right) + \frac{1}{2}(3\kappa_1 - cH)\\
&= \frac{H\Gamma}{2(4d_0\rho + H\Gamma)}(3\kappa_1 - cH) + \frac{\mu_0 d_0(4d_0\rho+H\Gamma)\rho}{2\mu_1}\Gamma\\
&\leq \frac{\kappa_1^2}{2cd_0\rho_*}\Gamma + \frac{\mu_0 d_0(4d_0\rho^* + 2\kappa_1/c)\rho^*}{2\mu_1}\Gamma\\
&= C_1\Gamma
\end{aligned}$$

for all $\Gamma \in (0, \Gamma_*]$, where we have set

$$C_1 := \frac{\kappa_1^2}{2cd_0\rho_*} + \frac{\mu_0 d_0(4d_0\rho^* + 2\kappa_1/c)\rho^*}{2\mu_1}. \tag{40}$$

It remains to observe that

$$\begin{aligned}
&f_1(H,\Gamma) + \frac{3\mu_0}{GH^3}(cH - \kappa_1)\\
&= \frac{6\mu_0}{GH^3}\left[\frac{\mu_0 d_0(4d_0\rho+H\Gamma)\rho}{2\mu_1\Gamma^2}\left(1 - \sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{3\kappa_1 - cH}{(4d_0\rho + H\Gamma)^2}\Gamma^2}\right) + \frac{1}{2}(3\kappa_1 - cH)\right]
\end{aligned}$$

and to set $C := 6\mu_0 c^3 C_1/G\kappa_1^3$, where C_1 is the constant given in (40), to complete the proof. $\square$

3.13 Corollary
There exist $\Gamma_* \in (0,1)$ and $C > 0$, both independent of (H,Γ), such that

$$-\frac{3\mu_0 c^3}{G\kappa_1^3}(cH-\kappa_1) < f_1(H,\Gamma) \leq -\frac{3\mu_0 c^3}{8G\kappa_1^3}(cH-\kappa_1) + C\Gamma$$

for all $(H,\Gamma) \in (\kappa_1/c, 2\kappa_1/c) \times (0,\Gamma_*]$.

Proof. This follows immediately from Lemma 3.9 and Lemma 3.12. $\square$

3.14 Theorem
Let $(H,\Gamma) \in C^2((\xi_-,\xi_+),\mathbb{R}^2)$ be the nonextendable solution to (33)–(34), together with the initial conditions (38). Then the ω-limit set of any orbit in the open positive invariant rectangle R is given by

$$\omega(H_0,\Gamma_0) = \{y_\omega\}, \quad (H_0,\Gamma_0) \in R.$$

Proof. (i) We start by proving that $\xi_+ = \infty$. For this purpose note that twice applying the Theorem of de l'Hôspital yields

$$\lim_{\Gamma\to 0} \frac{f_2(H,\Gamma)}{\Gamma} = -\frac{3\kappa_1 - cH}{2d_0 H} \geq -\frac{c}{d_0} =: -c_1$$

for all $H \in (\kappa_1/c, 2\kappa_1/c)$. Moreover, since $f_2(H,\Gamma)/\Gamma$ is continuous for all $\Gamma \in [0,1]$, there exists a $\overline{\Gamma} > 0$ such that

$$\lim_{\Gamma\to 0} \frac{f_2(H,\Gamma)}{\Gamma} \geq -2c_1, \quad \Gamma \in [0,\overline{\Gamma}].$$

Finally, for $\Gamma \in [\bar{\Gamma},1]$, we prove that there is a positive constant $c_g > 0$ such that $g(H,\Gamma) \leq c_g$ for all $H \in (\kappa_1/c, 2\kappa_1/c)$. Indeed, using that $0 \leq \rho \leq c_\rho$ for all $\Gamma \in [\bar{\Gamma},1]$, it holds that

$$\begin{aligned} g(H,\Gamma) &= \frac{\mu_0(4d_0\rho + H\Gamma)\rho}{2\mu_1\Gamma}\left(\sqrt{1 + 8\frac{\mu_1}{\mu_0}\frac{\Gamma^2(3\kappa_1 - cH)}{(4d_0\rho + H\Gamma)^2}} - 1\right) \\ &\leq \frac{\mu_0\rho}{2\mu_1}\sqrt{8\frac{\mu_1}{\mu_0}(3\kappa_1 - cH)} \\ &\leq \frac{\mu_0 c_\rho}{2\mu_1}\sqrt{8\frac{\mu_1}{\mu_0}} =: c_g. \end{aligned}$$

This leads us to the estimate

$$\frac{f_2(H,\Gamma)}{\Gamma} = -\frac{1}{H\Gamma}g(H,\Gamma) \geq -\frac{1}{H\Gamma}c_g \geq -\frac{cc_g}{\kappa_1\overline{\Gamma}} =: -c_2.$$

Defining $c^* := \max\{c_1, c_2\}$, we have proved that

$$\frac{f_2(H,\Gamma)}{\Gamma} \geq -c^*, \quad \Gamma \in [0,1].$$

From this inequality we may now derive the differential inequality

$$\Gamma'(\xi) \geq -c^*\Gamma(\xi), \quad \xi \in [0, \xi_+),$$

whereby we conclude that

$$\Gamma(\xi) \geq \Gamma_0 \exp(-c^*\xi), \quad \xi \in [0, \xi_+). \tag{41}$$

We know that if ξ_+ were finite then either Γ converges to the boundary of the phase space or it blows up. Inequality (42) and the fact that Γ is decreasing[5] (note that $f_2(H,\Gamma) < 0$ for all $(H,\Gamma) \in R$) rule out these two possibilities. Consequently, $\xi_+ = \infty$ and $\Gamma(\xi)$ converges to zero, as $\xi \to \infty$.
(ii) We show that there exists a constant $c_* > 0$ such that

$$\Gamma(\xi) \leq \Gamma_0 \exp(-c_*\xi), \quad \xi \in [0, \xi_+).$$

To this end let $\Gamma_* := \inf_{\xi \in [0,\xi_+)} \Gamma(\xi)$ and $\rho_* := \min_{\Gamma \in [\Gamma_*, \Gamma_0]}$. Then $\rho_* > 0$. We define the function

$$\tilde{g}(H,\Gamma) := \sqrt{(H\Gamma + 4d_0\rho)^2 + 8\frac{\mu_1}{\mu_0}(3\kappa_1 - cH)} - (H\Gamma + 4d_0\rho).$$

Note that $\tilde{g}(H,\Gamma) > 0$ for all (H,Γ) in $\tilde{R} := [\kappa_1/c, 2\kappa_1/c] \times [\Gamma_*, \Gamma_0]$ and set $\tilde{g}_* := \min_{(H,\Gamma)\in\tilde{R}} \tilde{g}(H,\Gamma)$. Then we have

$$-\frac{f_2(H,\Gamma)}{\Gamma} = \frac{\mu_0\rho}{2\mu_1 H\Gamma^2}\tilde{g}(H,\Gamma) \geq \frac{\mu_0\rho_* c}{4\mu_1\kappa_1\Gamma_0^2}\tilde{g}_* = c_* > 0$$

for all $(H,\Gamma) \in \tilde{R}$ with

$$c_* := \frac{\mu_0\rho_* c}{4\mu_1\kappa_1\Gamma_0^2}\tilde{g}_*.$$

From this inequality we may derive the differential inequality

$$\Gamma'(\xi) \leq -c_*\Gamma(\xi), \quad \xi \in [0, \xi_+),$$

whereby we conclude that

$$\Gamma(\xi) \leq \Gamma_0 \exp(-c_*\xi), \quad \xi \in [0, \xi_+). \tag{42}$$

(iii) It remains to prove that $\lim_{\xi\to\infty} H(\xi) = \kappa_1/c$. Let (H,Γ) be the solution to $(H,\Gamma)' = (f_1(H,\Gamma), f_2(H,\Gamma))$ with the initial datum (H_0,Γ_0). We already know from (i) that this

[5] Note that Lemma 3.2 treats global solutions, whereas here we deal only with nonextendable solutions which might not be global. The proof of Lemma 3.2 is based on the assumption that $\xi_- = -\infty$.

solution exists for all positive times and (ii) entails that $\Gamma(\xi)$ decays exponentially fast to zero as $\xi \to \infty$. Thus there exists $\xi_* > 0$ such that

$$\Gamma(\xi) \in (0, \Gamma_*] \quad \text{for all} \quad \xi \geq \xi_*.$$

Set $h_* := H(\xi_*) - \kappa_1/c$ and denote by $\underline{h}$ and by $\overline{h}$ the solutions of

$$\underline{h}'(\xi) = -\frac{3\mu_0 c^4}{G\kappa_1^3}\underline{h}(\xi) \quad \text{on} \quad (\xi_*, \infty), \qquad \underline{h}(\xi_*) = h_*$$

and

$$\overline{h}'(\xi) = -\frac{3\mu_0 c^4}{8G\kappa_1^3}\overline{h}(\xi) + \frac{C}{c}\Gamma(\xi) \quad \text{on} \quad (\xi_*, \infty), \qquad \overline{h}(\xi_*) = h_*,$$

respectively. Since Γ decays exponentially fast to zero as $\xi \to \infty$, both $\underline{h}$ and $\overline{h}$ decay exponentially fast to zero as $\xi \to \infty$. By Corollary 3.13 and the comparison principle for ODEs we have that

$$\underline{h}(\xi) \leq H(\xi) - \frac{\kappa_1}{c} \leq \overline{h}(\xi), \quad \xi \in [\xi_*, \infty).$$

This yields the assertion. $\square$

3.4 Study of the α-limit set $\alpha(H, \Gamma)$

In a last step we apply the Theorem of Grobmann-Hartmann to the equilibrium point $(2\kappa_1/c, 1)$ in order to guarantee the existence of a heteroclinic orbit which connects the steady states $(2\kappa_1/c, 1)$ and $(\kappa_1/c, 0)$. In order to guarantee that y_α is a hyperbolic critical point we need the additional property

$$\rho'(1) < 0, \tag{43}$$

where ρ is the extension of $-1/\sigma'$ as introduced above. Note that this condition is satisfied for both Szyszkowski's and Frumkin's law (see (39)), as in both cases we have $\rho'(1) = -1/\sigma_1 < 0$.

3.15 Theorem (Existence of a travelling wave)
There exists a heteroclinic orbit of the flow generated by $f = (f_1, f_2)$ which connects $(2\kappa_1/c, 1)$ and $(\kappa_1/c, 0)$.

Proof. (i) Note that

$$Df(y_\alpha) = \begin{pmatrix} -\frac{3\mu_0}{4G\kappa_1^3} & -\frac{3\mu_0^2 d_0 \rho'(1)}{4\mu_1 Gc\kappa_1^2}\left(\sqrt{1 + 2\frac{\mu_1 c^2}{\mu_0\kappa_1^2}} - 1\right) \\ 0 & -\frac{\mu_0\rho'(1)}{2\mu_1}\left(\sqrt{1 + 2\frac{\mu_1 c^2}{\mu_0\kappa_1^2}} - 1\right) \end{pmatrix}$$

is the Jacobian matrix of f in y_α. Then

$$(\lambda_s, v_s) := \left(-\frac{3\mu_0}{4G\kappa_1^3}, v_s\right) \quad \text{and} \quad (\lambda_u, v_u) := \left(-\frac{\mu_0\rho'(1)}{2\mu_1}\left(\sqrt{1 + 2\frac{\mu_1 c^2}{\mu_0\kappa_1^2}} - 1\right), v_u\right)$$

are the stable and unstable eigenpair of $Df(y_\alpha)$, respectively. In particular we have that y_α is a hyperbolic critical point, as there the linearisation of the vector field has no imaginary eigenvalues. Moreover, we introduce the notation

$$E_s := \mathrm{span}(v_s) \quad \text{and} \quad E_u := \mathrm{span}(v_u).$$

(ii) Denoting by φ the flow induced by f we invoke the Theorem of Grobmann–Hartmann (c.f. [1, Thm. 19.9]) to obtain that $\varphi|y_\alpha$ and the linearised flow $\exp^{(t\,Df(y_\alpha))}|0$ are isochronous flow equivalent. This particularly implies the existence of an $\varepsilon > 0$, a neighbourhood V of y_α, as well as a diffeomorphism $\phi \in \mathrm{Diff}^2(\varepsilon\mathbb{B}_0, V)$ such that, if $||v_u||_2 < \varepsilon$, then

$$\mathcal{C}_u := \{\phi(\tau v_u);\ |\tau| < 1\}$$

is a C^2-curve with $\mathcal{C}_u \subset M_u(y_\alpha)$, where $M_u(y_\alpha)$ is the unstable manifold through y_α.
(iii) Let now $T_{y_\alpha}(\mathcal{C}_u)$ be the tangential space of $\mathcal{C}_u$ (or $M_u(y_\alpha)$) at y_α. From [1, Thm. 19.11] we deduce that

$$T_{y_\alpha}(\mathcal{C}_u) = y_\alpha + E_u.$$

In particular this implies that

$$R \cap \mathcal{C}_u \neq \emptyset.$$

Given $y_0 \in R \cap \mathcal{C}_u$, let $\varphi(\cdot, y_0)$ be the solution of

$$(H', \Gamma') = f(H, \Gamma), \quad (H(0), \Gamma(0)) = y_0,$$

and $M_s(y_\omega)$ the stable manifold through y_ω. Then the fact that $y_0 \in R \cap M_s(y_\omega)$ implies that

$$\xi_+(y_0) = \infty \quad \text{and} \quad \lim_{\xi \to \infty} \varphi(\xi, y_0) = y_\omega. \tag{44}$$

Moreover, as $y_0 \in \mathcal{C}_u \subset M_u(y_\alpha)$, we additionally have that

$$\xi_-(y_0) = -\infty \quad \text{and} \quad \lim_{\xi \to -\infty} \varphi(\xi, y_0) = y_\alpha.$$

□

We finally use Theorem 3.15 to construct a travelling wave solution for the system (21)–(22) of partial differential equations. To this end let y_0 be as in the proof of Theorem 3.15 and let $(H, \Gamma) = \varphi(\cdot, y_0)$ be the solution of (44). We then introduce a pair of x-dependent functions by setting

$$\big(h_0(x), \gamma_0(x)\big) := \big(H(\xi), \Gamma(\xi)\big)_{t=0} = \big(H(x - ct), \Gamma(x - ct)\big)_{t=0} = \big(H(x), \Gamma(x)\big)$$

for $x \in \mathbb{R}$. From the above reasoning we know that $(h_0, \gamma_0) \in C^2(\mathbb{R}, \mathbb{R}^2)$ with

$$h_0, \gamma_0 > 0, \quad \frac{\kappa_1}{c} < h_0 < \frac{2\kappa_1}{c}, \quad \gamma_0 < 1$$

and the according system (21)–(22) of partial differential equations with initial conditions

$$h(0, x) = h_0(x), \quad \gamma(0, x) = \gamma_0(x)$$

possesses a travelling wave solution of the form

$$h(t, x) := h_0(x - ct), \quad \gamma(t, x) := \gamma_0(x - ct)$$

for all $(t, x) \in \mathbb{R}^2$.

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