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Structured Interior Point SQP Methods in Optimal Control

Trajectory optimization is an important means to improve performance in many industrial and scientific processes such as robotics or aerospace applications. A combination of SQP methods with direct collocation or multiple shooting discretizations has proven very successful for such problems. In realistic applications, however, very large QP subproblems with thousands of constraints may arise. The paper presents an approach based on highly efficient solution of the block-sparse QP by a structured interior point method.

1. Introduction

The first direct multiple shooting method for optimal control problems was developed by BOCK and PLITT [2,7] in 1983. Their code MUSCOD was successfully applied, e.g., in robot trajectory optimization [5,6,10]. Other direct SQP methods based on collocation have been developed in the sequel [1,3,11]. However, the solution of QP subproblems becomes a bottleneck in large scale applications, in specific when fine discretizations are needed. Exploitation of sparseness is then crucial. But general-purpose sparse solvers have several drawbacks: Analyzing the sparseness pattern is expensive, the fill-in is unknown a priori, and reasons for failure of a factorization may be hard to detect. Therefore we have developed a structured approach for control problems, based on the following general strategy. The nonlinear problem is set up in a way that reduces nonlinear coupling. Usually this means introducing additional variables and conditions, but it also gives the problem a clearer structure. The surplus variables and conditions are then first eliminated from the *linearized* problem, which shifts computational effort from "difficult" nonlinear tasks to "easy" linear ones and increases efficiency, at least on hard problems. This strategy leads to specific formulations of continuous and discrete control problems on the two upper levels in our direct approach. The resulting *multi-stage block-sparse* structure is then exploited in the solution of QP and KKT subproblems on the two lower levels.

In the present paper we outline the essentials of the structured approach, which was presented in [8,9] along with a detailed description of the highly efficient recursive multi-stage KKT solver MSKKT. First computational results for the primal-dual multi-stage interior point QP solver MSIPM are then given.

2. Structured direct SQP approach

In this section we sketch the four levels of our structured approach. On the first level we consider optimal control problems in a standard form on unit interval $I = [0,1]$:

$$\text{Minimize} \quad \phi(x(1)) \qquad \text{subject to} \qquad \left\{ \begin{array}{r} \dot{x} - f(x,u) = 0 \\ g(x,u) \geq 0 \end{array} \right\} \text{ on } I, \quad r(x(t_1), \ldots, x(t_k)) = 0. \tag{1}$$

Our structural assumption here is that the multipoint boundary conditions at times $0 = t_1 < \cdots < t_k = 1$ are *second order decoupled*, admitting a partitioning into completely separated conditions r^0 and linearly coupled conditions r^1,

$$r(x_1, \ldots, x_k) = \begin{pmatrix} r^0(x_1, \ldots, x_k) \\ r^1(x_1, \ldots, x_k) \end{pmatrix}, \qquad \begin{array}{l} r^0(x_1, \ldots, x_k) = \left(r_1^0(x_1)^T, \ldots, r_k^0(x_k)^T \right)^T, \\ r^1(x_1, \ldots, x_k) = r_1^1(x_1) + \cdots + r_k^1(x_k). \end{array}$$

Dummy variables may always be used to separate nonlinearly coupled boundary conditions if necessary. Many other generalizations can also be reduced to standard form (1) by suitable transformations [9].

To discretize (1) by the direct boundary value problem approach we choose a grid $0 = \tau_1 < \cdots < \tau_m = 1$ as a refinement of $\{t_i\}_{i=1}^k$, with grid inclusion σ, $t_i = \tau_{\sigma(i)}$. A finite-dimensional piece-wise control representation is now defined by free control parameters u_j and fixed base functions v_j with local support, $u(t) = v_j(t, u_j)$ on $I_j = [\tau_j, \tau_{j+1})$. Control jumps are permitted at the nodes τ_j.

A state discretization by *collocation* uses polynomials p_j of degree κ_j as local representations of the trajectory x on I_j. The polynomial $p_j(t, x_j, z_j)$ is uniquely parameterized by its initial value $x_j \equiv p_j(\tau_j, x_j, z_j)$ and derivatives $z_j^i \equiv \dot{p}_j(\tau_j^i, x_j, z_j)$ at *collocation points* $\tau_j^i = \tau_j + \rho_j^i(\tau_{j+1} - \tau_j)$, $0 \leq \rho_j^1 < \cdots < \rho_j^{\kappa_j} \leq 1$. Polynomials p_j are required to satisfy the differential equation at all collocation points, specified by *collocation conditions*

$$c_j^i(x_j, z_j, u_j) \triangleq z_j^i - f(p_j(\tau_j^i, x_j, z_j), v_j(\tau_j^i, u_j)) = 0, \qquad j = 1(1)m-1, \quad i = 1(1)\kappa_j.$$

Global continuity of the piece-wise trajectory representation is ensured by *connection conditions*

$$h_j(x_j, z_j, x_{j+1}) \triangleq p_j(\tau_{j+1}, x_j, z_j) - x_{j+1} = 0, \qquad j = 1(1)m-1.$$

The structural assumption here lies in the form of the connection condition which equals x_{j+1} to a (nonlinear) function of *local* variables in stage j only. This does not restrict the class of admissible collocation schemes: the assumption can always be satisfied by a suitable formulation of collocation and connection conditions. Moreover, it is automatically satisfied for multiple shooting or finite difference discretizations which may be used alternatively.

In order to formulate the discrete control problem we define variable vectors $y_j \triangleq (x_j, z_j, u_j)$, $y \triangleq (y_1, \ldots, y_m)$ and the objective function $F_1(y) \triangleq \phi(x_m)$, and collect equality and inequality constraints

$$F_2(y) \triangleq \begin{pmatrix} \{c_j(y_j)\}_{j=1}^{m-1} \\ \{h_j(y_j, x_{j+1})\}_{j=1}^{m-1} \\ r(x_1, \ldots, x_m) \end{pmatrix}, \qquad F_3(y) \triangleq \left(\{g_j(y_j)\}_{j=1}^{m}\right),$$

where $r(x_1, \ldots, x_m)$ replaces $r(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for notational simplicity, and $g_j(y_j) \triangleq g(x_j, v_j(\tau_j, u_j))$. The discretized control problem on the second level is now written as a general nonlinear programming problem (NLP):

$$\text{Minimize} \quad F_1(y) \qquad \text{subject to} \qquad F_2(y) = 0, \quad F_3(y) \geq 0. \tag{2}$$

For the numerical solution of (2) we apply an SQP iteration, $y^{k+1} = y^k + \alpha^k \Delta y^k$, $\alpha^k \in (0, 1]$, where each search direction Δy^k is obtained as solution of a linear-quadratic subproblem (QP) on the third level:

$$\text{Minimize} \quad \frac{1}{2}\Delta y^T H^k \Delta y + J_1^k \Delta y \qquad \text{subject to} \qquad J_2^k \Delta y + F_2^k = 0, \quad J_3^k \Delta y + F_3^k \geq 0. \tag{3}$$

Here we define $F_i^k \triangleq F_i(y^k)$, $J_i^k \triangleq F_i'(y^k)$, and $H^k \approx L_{yy}(y^k, w_2^k, w_3^k)$ approximates the Hessian of the Lagrangian $L(y, w_2, w_3) \triangleq F_1(y) - w_2^T F_2(y) - w_3^T F_3(y)$. The m-stage block-sparse QP structure will now be established.

Theorem 1. *The Hessian H^k and Jacobians J_2^k, J_3^k are block-diagonal, except for superdiagonal blocks $-I$ and a full row of blocks in J_2^k which are produced by the connection and boundary conditions h_j and r^1, respectively.*

Proof. The statement on the Jacobians is obviously true provided that the component functions of F_2, F_3 are suitably arranged (c_j, h_j, r_j^0 must be combined in each stage). Since ϕ, c_j, h_j, r, g_j are all second order decoupled, the Hessians of their components are block-diagonal and so is H^k. $\square$

In collocation we apply a local condensing algorithm to eliminate all the collocation variables z_j from the QP using the linearized collocation conditions. The multi-stage QP structure remains unchanged, except that collocation variables and conditions are now absent as in a multiple shooting or finite difference discretization. The resulting QP subproblems within the major SQP iteration are again solved iteratively, either by an *active set strategy (ASS)* or an *interior point method (IPM)*. In both cases, each step of the minor iteration requires the solution of a symmetric, indefinite system of linear equations which represents the Karush-Kuhn-Tucker optimality conditions of a purely equality-constrained subproblem. These KKT systems on the fourth, lowest level are basically of the form

$$\Omega z + a = \begin{pmatrix} H & C^T \\ C & \end{pmatrix} \begin{pmatrix} \delta y \\ -\delta w \end{pmatrix} + \begin{pmatrix} f \\ c \end{pmatrix} = 0, \tag{4}$$

where C combines J_2 with active rows of J_3 (ASS), or $C = J_2$ and H is modified by barrier terms (IPM, see below). In addition to this optimization superstructure the KKT matrix Ω possesses a substructure according to Theorem 1.

3. Structured interior point method

Interior point or barrier methods became popular in *linear programming* after the appearance of KARMARKAR's work [4] in 1984. A vast number of papers on this topic has been published since, including extensions to convex and quadratic programming. Among the few ones specifically dedicated to discrete control problems are [8,9,12]. Because of the typically large number of inequality constraints we advocate an interior point method in this context, in specific a primal-dual one which uses dual information from the major SQP iteration for a warm start strategy.

However, active set strategies are expected to need less minor iterations in the final SQP iterations or when improving an optimal solution by grid refinement.

In the following we give a short outline of our interior point approach for quadratic problems. Let us consider the general QP (3) again, but drop superscript k and replace Δy by y. After introducing slacks s we obtain the corresponding barrier problem with barrier parameter $\mu > 0$,

$$\text{Minimize} \quad \frac{1}{2} y^T H^k y + J_1^k y - \mu \sum_i \ln s_i \qquad \text{subject to} \qquad \left\{ \begin{array}{r} J_2 y + F_2 = 0 \\ -s + J_3 y + F_3 = 0 \end{array} \right\}. \tag{5}$$

Assume that the QP (3) has a unique solution. Then the solutions $(y_\mu, s_\mu, w_{2\mu}, w_{3\mu})$ of (5) are also unique, depend analytically on μ, and converge to the QP solution along the primal-dual *central path* as $\mu \longrightarrow 0$. The solutions are characterized by a vanishing gradient of the Lagrangian of (5), yielding first order optimality conditions

$$Hy + J_1^T - J_2^T w_2 - J_3^T w_3 = 0, \qquad SW_3 e - \mu e = 0, \qquad J_2 y + F_2 = 0, \qquad J_3 y + F_3 - s = 0. \tag{6}$$

Here $S \triangleq \operatorname{diag}(\{s_i\})$, $W_3 \triangleq \operatorname{diag}(\{w_{3i}\})$ and $e \triangleq (1, \ldots, 1)$, as usual in this context. From (6) it is apparent that the solutions of (5) are all primal and dual feasible, and the duality gap is given by $s^T w_3 = e^T SW_3 e = n_3 \mu > 0$. Efficient interior point methods reduce μ after each Newton iteration on (6), thus working on feasibility, centrality and optimality simultaneously. The Newton increment $(\delta y, \delta s, \delta w_2, \delta w_3)$ is given as solution of the KKT system

$$\left[\begin{array}{cc|cc} H & & J_2^T & J_3^T \\ & \Theta & & -I \\ \hline J_2 & & & \\ J_3 & -I & & \end{array}\right] \left[\begin{array}{c} \delta y \\ \delta s \\ \hline -\delta w_2 \\ -\delta w_3 \end{array}\right] + \left[\begin{array}{l} Hy + J_1^T - J_2^T w_2 - J_3^T w_3 \\ \theta \\ \hline J_2 y + F_2 \\ J_3 y + F_3 - s \end{array}\right] = 0, \tag{7}$$

where $\Theta \triangleq S^{-1} W_3$ and $\theta \triangleq w_3 - \mu S^{-1} e$. We treat inequalities in the most general form, specified as lower and upper bounds $b_l \leq y \leq b_u$ and ranges $r_l \leq By \leq r_u$, which leads to a substructure in J_3 and associated quantities,

$$J_3 = \begin{pmatrix} I \\ -I \\ B \\ -B \end{pmatrix}, \quad F_3 = \begin{pmatrix} -b_l \\ b_u \\ -r_l \\ r_u \end{pmatrix}, \quad s = \begin{pmatrix} s_l \\ s_u \\ t_l \\ t_u \end{pmatrix}, \quad w_3 = \begin{pmatrix} u_l \\ u_u \\ v_l \\ v_u \end{pmatrix}, \quad \theta = \begin{pmatrix} \phi_l \\ \phi_u \\ \psi_l \\ \psi_u \end{pmatrix}, \quad \Theta = \operatorname{diag}(\Phi_l, \Phi_u, \Psi_l, \Psi_u).$$

Inserting this into (7) yields a very large and sparse KKT system in accordance with our general strategy. After eliminating most of the increments by simple calculations [9] we end up with the reduced KKT system

$$\left[\begin{array}{c|cc} H + \Phi & A^T & B^T \\ \hline A & & \\ B & & -\Psi^{-1} \end{array}\right] \left[\begin{array}{c} \delta y \\ \hline -\delta z \\ -\delta v_l \end{array}\right] + \left[\begin{array}{l} h + \Phi \beta_l + \phi \\ \hline a \\ \rho_l + \Psi^{-1} \psi \end{array}\right] = 0 \qquad (A \triangleq J_2, \quad z \triangleq w_2). \tag{8}$$

This system has the original size and sparse structure of QP (3), as determined by matrices A, B, H. Contributions from bounds and ranges are concentrated in the nonnegative diagonal matrices Φ and Ψ. A problem-specific KKT solver should now be employed. In the control context one might further reduce (8) to the form (4) without destroying the structure. But this is numerically unstable, so we apply a slightly modified version of MSKKT directly.

4. Computational results

Numerical comparisons are performed on a point-to-point trajectory optimization problem for an n-joint robot. Let $x, u, Du \in \mathbb{R}^n$ denote joint angles, motor torques, and joint torques, respectively. A simplified multibody model of the robot is then characterized by the symmetric, positive definite mass matrix $M(x)$, the quadratic centrifugal and Coriolis torques $C(x)(\dot{x}, \dot{x})$, and gravity torques $\gamma(x)$. The aim is to minimize the maneuver time T subject to

$$\begin{array}{rlll} M(x)\ddot{x} = C(x)(\dot{x}, \dot{x}) + \gamma(x) + Du & \text{on } [0, T], & x(0) = A, & x(T) = B, \\ |u^i| \leq u^i_{\max} & \text{on } [0, T], & \dot{x}(0) = 0, & \dot{x}(T) = 0. \end{array}$$

(For numerical computations the problem is transformed to a first order system on [0,1].) A certain maneuver is optimized by MUSCOD on different grids for a hypothetical 3-joint elbow robot and for a 6-joint industrial robot KUKA IR761. All the QP subproblems are written to data files, solved by MSIPM to an accuracy of 3 or 6 decimals, and average CPU times of the MUSCOD QP solver and of MSIPM are compared. Results are given in Table 1.

Table 1: Average CPU times of MUSCOD QP solver vs. MSIPM for various grid sizes

Time	3-joint elbow robot					6-joint KUKA IR761		
steps	MUSCOD	MSIPM 10^{-3}		MSIPM 10^{-6}		MUSCOD	MSIPM 10^{-3}	MSIPM 10^{-6}
10	0.05	[0.29]	0.12	[0.32]	0.17	0.27	0.37	0.47
20	0.21	[0.62]	0.22	[0.65]	0.29	1.37	0.80	0.97
40	1.34	[1.84]	0.60	[2.01]	0.70			
80	8.62	[4.59]	1.22	[4.69]	1.49			
160	79.41		2.77		3.24			

The MUSCOD QP solver eliminates all but the initial states by a condensing projection; its active set strategy then operates on the condensed QP. MSIPM solves the full-size structured QP. Since $\Delta y^k \to 0$ we always initialize $y = 0$ in the barrier problem but shift infeasible components inside their bounds. The initial slacks are $s = J_3 y + F_3$, with negative components shifted above 0. For the dual variables we employ a cold start heuristic in the first SQP iteration, trying to get near the central path. The warm start strategy simply uses w_2^k, w_3^k, the values of the previous major iteration. Some CPU times of MSIPM *without* the warm start strategy are given in brackets.

Results show that the MUSCOD QP solver performs better on small problems, but the break-even point is reached on relatively coarse grids. MSIPM is already significantly faster (even for 6 digits accuracy) on 20 and 40 time steps for the 6-joint and 3-joint robots, respectively. Following the columns for the 3-joint robot we observe that the MUSCOD QP solver exhibits cubic complexity $\mathcal{O}(m^3)$, while CPU times for MSIPM almost show the linear complexity $\mathcal{O}(m)$ of MSKKT—evidently the number of minor iterations grows very slowly with m. Although the asymptotic superiority of MSIPM can be attributed to the structure exploitation on the KKT level, the numbers in brackets indicate that the warm start strategy is also very important. It cuts down the average number of minor iterations by 47–73 %, with increasing savings on finer grids.

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