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ON A NEW MODEL FOR CONTINUOUS COALESCENCE AND BREAKAGE PROCESSES WITH DIFFUSION

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Abstract. We study a new model for the evolution of a liquid-liquid dispersion. The droplets of the dispersed phase are supposed to move due to diffusion and to undergo coalescence and breakage. The main feature of the model is the inclusion of a maximal droplet size. This requires a consistent mechanism opposing the increase of droplets due to coalescence. The resulting system of uncountably many coupled reaction-diffusion equations is interpreted as a vector-valued Cauchy problem. We prove existence and uniqueness of nonnegative and mass-preserving solutions. Furthermore, we give sufficient conditions for global existence.

1. Introduction

Since the pioneering work of von Smoluchowski [33, 34], various models for cluster growth and degradation have been developed and mathematically investigated. In most of those models a very large number of particles is considered, which are assumed to be completely identified by their size, like mass or volume. The particles then undergo the influences of coagulation and/or fragmentation, meaning that they can merge to build larger particles or split into several smaller ones. The reasons causing aggregation and degradation of particles depend on the physical context of these processes, which arise in a multitude of situations—for instance in astronomy, the oil industry, and polymer and aerosol sciences to name just a few. We refer to [16, 26] for surveys of the various models.

The aim of this paper is to discuss well-posedness of a new model describing the time evolution of a liquid-liquid dispersion. Such a dispersion is formed by two immiscible liquids, where one of them consists of a very large number of droplets that are finely distributed in the other one. These droplets diffuse according to Brownian motion and may then collide, producing either a larger droplet due to coalescence or several smaller droplets

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in the case of high-energy collisions. In addition, these droplets can spontaneously decay into smaller droplets as a result of external forces.

The main feature of the model considered in the present paper is that it pays attention to the facts that droplets cannot become arbitrarily large, and that experimental observations (see [31]) suggest the existence of a maximal droplet size $y_0 \in (0,\infty)$ beyond which no droplet can survive. In literature, such a maximal size for droplet masses has either been disregarded so far or has been introduced only as an artificial cut-off (cf. [40]) neglecting a fundamental inconsistency of the model. Indeed, imposing a maximal droplet mass requires a new interaction mechanism to prevent the occurrence of droplets with masses beyond y_0 resulting from coalescence. A particular mechanism—called (*volume*) *scattering*—has been proposed by Fasano and Rosso [17]. The idea is that two colliding droplets with cumulative mass exceeding the maximal mass y_0 can merge, but result in a highly unstable droplet, which instantaneously splits into several droplets all with mass within the admissible range $Y := (0, y_0]$. We take up this interaction mechanism in our equations.

To be more precise, let us denote by $u = u(y) = u(t,x,y)$ the droplet-size distribution function at time t and position x, where $y \in Y$ refers to the droplet size. Assuming the whole system under consideration to be isolated so that there is no heat or mass exchange, the evolution of the liquid-liquid dispersion can be described by the uncountable set of coupled reaction-diffusion equations

$$\begin{aligned} \partial_t u(y) - d(t,y)\Delta_x u(y) &= L(t,x,u)(y) && \text{in } \Omega, \quad t>0,\ y\in Y,\\ \partial_\nu u(y) &= 0 && \text{on } \partial\Omega,\ t>0,\ y\in Y, \\ u(0,\cdot,y) &= u^0(y) && \text{in } \Omega, \qquad\quad\ y \in Y, \end{aligned} \tag{$*$}$$

where $u^0 = u^0(x,y)$ is a given initial distribution. Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$, and ν denotes its outward unit normal vector.

Diffusion of the droplets is governed by the time-dependent coefficients $d(t,y) > 0$. It seems to be reasonable to assume that $d(t,\cdot)$ is a nonincreasing function of droplet size since intuition suggests that large droplets diffuse more slowly than small ones. Hence we do not significantly restrict the physical scope of applications if we suppose that $d(t,\cdot)$ is bounded from below by some strictly positive constant.

The reaction terms

$$L(t,x,u) := L_{\mathsf{b}}(t,x,u) + L_{\mathsf{c}}(t,x,u) + L_{\mathsf{s}}(t,x,u)$$

are defined by

$$
\begin{aligned}
L_{\mathsf{b}}(\cdot,\cdot,u)(y) &:= \int_y^{y_0} \gamma(y',y)u(y')\ \mathrm{d}y' - u(y)\int_0^y \frac{y'}{y}\gamma(y,y')\ \mathrm{d}y'\ , \\
L_{\mathsf{c}}(\cdot,\cdot,u)(y) &:= \frac{1}{2}\int_0^y K(y',y-y')P(y',y-y')u(y')u(y-y')\ \mathrm{d}y' \\
&\qquad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} K(y'',y'-y'')Q(y'',y'-y'')\beta_{\mathsf{c}}(y',y) \\
&\qquad\qquad\qquad \times u(y'')u(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\
&\qquad - u(y)\int_0^{y_0-y} K(y,y')\big\{P(y,y')+Q(y,y')\big\}u(y')\ \mathrm{d}y'\ , \\
L_{\mathsf{s}}(\cdot,\cdot,u)(y) &:= \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\beta_{\mathsf{s}}(y',y) \\
&\qquad\qquad\qquad \times u(y'')u(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\
&\qquad - u(y)\int_{y_0-y}^{y_0} K(y,y')u(y')\ \mathrm{d}y'\ ,
\end{aligned}
$$

for $y \in Y = (0, y_0]$.

The term $L_{\mathsf{b}}(t,x,u)(y)$ accounts for the gain and loss of droplets of mass y due to multiple spontaneous breakage, where $\gamma(y,y') = \gamma(t,x,y,y')$ represents the rate at which a droplet of mass $y \in Y$ splits into a droplet of mass $y' \in (0,y)$.

When two droplets y and y' with cumulative mass $y+y' \le y_0$ collide, one of the following three different events takes place. The droplets either coalesce with probability $P(t,x,y,y')$, or a shattering of these droplets occurs with probability $Q(t,x,y,y')$, or—for instance, in the case of grazing droplets—nothing happens, meaning that the droplets involved remain unchanged. By $K(t,x,y,y')$ we denote the collision rate, and $\beta_{\mathsf{c}}(t,x,y+y',y'')$ stands for the distribution function for droplets of mass $y'' \in (0,y+y')$, produced by the decay of a droplet of mass $y+y' \le y_0$ after a high-energy collision. One intuitively expects that collisional breakage preserves the total mass, which can be expressed by the formula

$$
\int_0^{y+y'} y''\beta_{\mathsf{c}}(t,x,y+y',y'')\ \mathrm{d}y'' = y+y'\ , \quad (t,x)\in\mathbb{R}^+\times\Omega\ , \quad y+y'\in Y\ .
$$

Moreover, consistency of the model demands

$$
0 \le P(t,x,y,y') + Q(t,x,y,y') \le 1\ , \quad (t,x)\in\mathbb{R}^+\times\Omega\ , \quad y,y'\in Y\ , \tag{1.1}
$$

and

$$R(t,x,y,y') = R(t,x,y',y) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad y,y' \in Y \ ,$$

for $R \in \{K,P,Q\}$. The formation and depletion of droplets of mass y, owing to the above three events, are expressed by $L_{\mathsf{c}}(t,x,u)(y)$.

Finally, the scattering term $L_{\mathsf{s}}(t,x,u)(y)$ represents the interaction of two colliding droplets y and y' with cumulative mass $y+y' > y_0$ immediately scattering into several droplets all with mass in $Y = (0, y_0]$. The resulting daughter droplets $y'' \in Y$ are distributed according to the function $\beta_{\mathsf{s}}(t,x,y+y',y'')$. Again one expects that this interaction conserves the mass; that is,

$$\int_0^{y_0} y''\beta_{\mathsf{s}}(t,x,y+y',y'')\mathrm{d}y'' = y+y', \ (t,x) \in \mathbb{R}^+ \times \Omega, \ y+y' \in (y_0, 2y_0].$$

All factors 1/2 in the definitions of L_{c} and L_{s} come in to compensate for double counting.

Our model extends the previously mentioned model for nondiffusive processes proposed in [17]. Besides taking into account Brownian movement, our equations also include multiple breakage and collisional fragmentation. Formally, the model considered in [17] can be derived from our model by putting $d \equiv 0$ and $P \equiv 1$ (implying $Q \equiv 0$ according to (1.1)), and assuming droplets to decay into exactly two daughter droplets. In addition to the just-cited paper we refer to [11, 18, 19, 29] for more information concerning the physical background, mathematical results, and numerical simulations.

Let us point out that a broadly based mathematical treatment of collisional breakage is apparently lacking. Even though contemplated in physical literature (see [12, 13, 45]), only a few mathematical results [27, 41] on this issue are published, at least to the best of our knowledge.

Well-posedness of the spatially homogeneous version of $(*)$ is studied in [41] by the author (there the case $Q \equiv 1 - P$ is treated), whereas certain aspects of asymptotic behaviour are investigated in [43].

Contrary to the spatially homogeneous coagulation-fragmentation equations, where literature is quite extensive, less seems to be known for the case including diffusion. Most of the research on diffusive processes is dedicated to its discrete version, where the particle size y ranges in the set of positive integers and the integrals are replaced by sums (or series). We refer to [25, 28] and the references therein for more detailed information about this case. Note that these two papers, and also the references therein, consider neither collisional breakage nor a maximal particle size; thus, there occurs no scattering as well. Concerning results for the situation including both of these processes, that is, for the discrete version of $(*)$, we refer to [44].

The number of articles on diffusive coagulation-fragmentation equations shrinks even more if the processes under consideration are assumed to be continuous. For this case, only a few papers are available—but again, all of them take into account neither a maximal droplet size nor high-energy collisions. The classical coagulation-fragmentation equations considered in these papers are formally derived from $(*)$ by putting $y_0 := \infty$ and $P \equiv 1$.

The first paper treating continuous coagulation and fragmentation processes with diffusion was written by Amann [6]. There, the author studies the situation $\Omega = \mathbb{R}^n$ so that, in particular, there are no boundary conditions. However, allowing more general diffusion operators (and including also convection), the resulting equations are shown to admit a unique local strong solution. Global existence is obtained either if space dimension equals 1, if diffusion is independent of droplet size, or if no coagulation occurs, that is, if the equations are linear. The approach chosen in this paper is to interpret the problem as a Banach-space-valued Cauchy problem (see below).

This idea is taken up in the paper of Amann and Weber [10] in order to investigate the behaviour of particles suspended in a carrier fluid leading to an additional coupling of the coagulation-fragmentation equations with the Navier-Stokes equation. Again, well-posedness locally in time is shown for $\Omega = \mathbb{R}^n$.

A completely different approach is chosen by Laurençot and Mischler [24]. Based on weak and strong compactness methods in L_1, they prove global existence (but not uniqueness) of weak solutions in case of binary fragmentation and under suitable assumptions on the structure of the kernels. They also consider large-time behaviour of solutions in particular situations. Some of these results on global existence have been subsequently improved by Mischler and Rodriguez-Ricard [30].

A further result on diffusive continuous coagulation-fragmentation equations is due to Deaconu and Fournier [14], who use probabilistic methods to prove global existence of "measure solutions."

Let us now briefly outline the contents of this paper. Our approach to problem $(*)$ is to handle it as an abstract vector-valued Cauchy problem of the form

$$\dot{u} + A(t)u = L(t,u)\ , \quad t > 0\ , \qquad u(0) = u^0\ .$$

Formally, this is obtained by putting $A(t) := -d(t,\cdot)\Delta$ with respect to Neumann boundary conditions. It turns out that, as in the scalar case, the operator $-A(t)$ is the generator of an analytic semigroup on $L_p(\Omega,E)$ with domain of definition

$$D\big(A(t)\big) \doteq H^2_{p,\mathcal{B}}(\Omega,E) := \big\{u \in H^2_p(\Omega,E)\,;\, \partial_\nu u = 0\big\}\ ,$$

where E is an appropriate function space over Y. This nontrivial generation result is due to a recent paper of Denk, Hieber, and Prüss [15] and requires a so-called UMD space E. This rules out the physically reasonable choice $E = L_1(Y)$ as state space. Furthermore, once it is made rigorous that $-A(t)$ generates an analytic semigroup, it is indispensable to have precise information about the interpolation spaces between $L_p(\Omega,E)$ and $D\big(A(t)\big) \doteq H^2_{p,\mathcal{B}}(\Omega,E)$ in order to take full advantage of semigroup theory. Such a characterization requires a Hilbert space E leading to the state space $E = L_2(Y)$.

In the next section we introduce some notation and state the required result on interpolation with boundary conditions. Section 3 deals with the generator $-A(t)$ and lists some of its useful properties. In Section 4 we give a precise description of how we will interpret the reaction terms. In the final section, Section 5, we focus on well-posedness of problem $(*)$. We prove existence and uniqueness of nonnegative and mass-preserving solutions. Moreover, we also derive sufficient conditions for global existence.

We conclude the introduction with a summary of our main results. They will be stated in a more precise manner in Section 5, where their proofs will also be given.

Theorem 1.1. *Let all of the kernels be nonnegative and sufficiently smooth. Suppose that* $\max\{1, n/4\} < p < \infty$. *Then, for each* $u^0 \geq 0$ *satisfying*

$$\int_\Omega \Big(\int_Y \big|\partial_x^\alpha u^0(x,y)\big|^2\ \mathrm{d}y\Big)^{p/2}\ \mathrm{d}x < \infty\ , \quad |\alpha| \leq 2,$$

and $\partial_\nu u^0(x,y) = 0$, $(x,y) \in \partial\Omega \times Y$, *there exists* $T > 0$ *such that problem* $(*)$ *admits a unique nonnegative solution* u *on* $[0,T)$. *If collisional breakage and scattering are mass-preserving, then*

$$\int_\Omega \int_Y y\, u(t,x,y)\ \mathrm{d}y\mathrm{d}x = \int_\Omega \int_Y y\, u^0(x,y)\ \mathrm{d}y\mathrm{d}x\ , \quad 0 \leq t < T\ .$$

In addition, $T = \infty$ *either if* $n < 2p$ *and diffusion is independent of time and droplet size or if* $n = 1$ *and collisional breakage and scattering are binary processes.*

We refer to Example 5.13 for physically relevant kernels satisfying the assumptions required.

2. Preliminaries

We briefly collect some basic spaces and their properties, which we will use in the sequel. For more detailed information and proofs we refer in particular to [5], but also to [4, 9].

In this paper, all vector spaces are over the reals. If there are implicit or explicit references to complex numbers in a given formula, then it is understood that the latter is interpreted as the corresponding complexification.

Let X and Z be locally convex spaces. We denote by $\mathcal{L}(X,Z)$ the set of all bounded linear operators from X into Z, and we put $\mathcal{L}(X) := \mathcal{L}(X,X)$. If X is a linear subspace of Z such that the natural injection $i : [x \mapsto x]$ belongs to $\mathcal{L}(X,Z)$, we express this by $X \hookrightarrow Z$. By $X \doteq Z$ we mean $X \hookrightarrow Z$ and $Z \hookrightarrow X$.

Let E be an arbitrary Banach space and Ω a nonempty open subset of $\mathbb{R}^n$. For $s \in \mathbb{R}^+$, $C^s(\Omega,E)$ $\big[$respectively $BUC^s(\Omega,E)\big]$ consists of all functions $u : \Omega \to E$ having $\big[$bounded and uniformly continuous$\big]$ derivatives of orders at most $[s]$ and whose derivatives of order $[s]$ are $\big[$uniformly$\big]$ Hölder continuous of exponent $s-[s]$, if $s \notin \mathbb{N}$. The space $BUC^\infty(\Omega,E)$ has then an obvious meaning. $C_c(\Omega)$ stands for the space of all continuous functions $u : \Omega \to \mathbb{R}$ with compact support.

We denote by $\mathcal{D}(\Omega,E)$ the space of all E-valued test functions on Ω, and we put $\mathcal{D}(\Omega) := \mathcal{D}(\Omega,\mathbb{R})$. Then $\mathcal{D}'(\Omega,E) := \mathcal{L}\big(\mathcal{D}(\Omega),E\big)$ is the space of all E-valued distributions on Ω.

$W_p^s(\Omega,E)$ is the usual Sobolev-Slobodeckii space on Ω of order $s \in \mathbb{R}$ and integrability index $p \in [1,\infty]$. The Bessel potential space of order $s \in \mathbb{R}$ and integrability index $p \in (1,\infty)$ is denoted by $H_p^s(\mathbb{R}^n,E)$. We write $B_{p,q}^s(\mathbb{R}^n,E)$ for the Besov space, where $s \in \mathbb{R}$ and $p,q \in [1,\infty]$. All of the spaces $W_p^s(\Omega,E)$, $H_p^s(\mathbb{R}^n,E)$, and $B_{p,q}^s(\mathbb{R}^n,E)$ are equipped with their usual norms. Then it is known that for $1 \le p, p_i, q, q_i \le \infty$ and $s, s_i \in \mathbb{R}$

$$B_{p,q_1}^s(\mathbb{R}^n,E) \hookrightarrow B_{p,q_0}^s(\mathbb{R}^n,E)\ , \quad 1 \le q_1 \le q_0 \le \infty\ , \tag{2.1}$$

$$B_{p,q_1}^{s_1}(\mathbb{R}^n,E) \hookrightarrow B_{p,q_0}^{s_0}(\mathbb{R}^n,E)\ , \quad s_1 > s_0\ , \tag{2.2}$$

and

$$B_{p_1,q}^{s_1}(\mathbb{R}^n,E) \hookrightarrow B_{p_0,q}^{s_0}(\mathbb{R}^n,E)\ , \quad p_1 < p_0\ , \quad s_1 - n/p_1 \ge s_0 - n/p_0\ . \tag{2.3}$$

For the remainder, let E be a UMD space (see [4, III. Section 4.4]). Then

$$H_p^m(\mathbb{R}^n,E) \doteq W_p^m(\mathbb{R}^n,E)\ , \quad m \in \mathbb{Z}\ , \quad 1 < p < \infty\ , \tag{2.4}$$

and

$$B_{p,p}^s(\mathbb{R}^n,E) \doteq W_p^s(\mathbb{R}^n,E)\ , \quad s \in \mathbb{R} \setminus \mathbb{Z}\ , \quad 1 \le p < \infty\ . \tag{2.5}$$

Moreover, denoting for $0 < \theta < 1$ and $1 \leq q \leq \infty$ by $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor of exponent θ and parameter q and by $[\cdot,\cdot]_\theta$ the complex interpolation functor of exponent θ, we have, for $1 \leq p, q, q_i \leq \infty$ and $s_i \in \mathbb{R}$ with $s_0 \neq s_1$,

$$\big(B^{s_0}_{p,q_0}(\mathbb{R}^n,E), B^{s_1}_{p,q_1}(\mathbb{R}^n,E)\big)_{\theta,q} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n,E) \tag{2.6}$$

as well as

$$\big[B^{s_0}_{p,q}(\mathbb{R}^n,E), B^{s_1}_{p,q}(\mathbb{R}^n,E)\big]_{\theta} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n,E)\ , \quad q < \infty\ . \tag{2.7}$$

Observe that (2.5)–(2.7) remain valid for arbitrary Banach spaces E. Furthermore,

$$\big[H^{s_0}_{p}(\mathbb{R}^n,E), H^{s_1}_{p}(\mathbb{R}^n,E)\big]_{\theta} \doteq H^{(1-\theta)s_0+\theta s_1}_{p}(\mathbb{R}^n,E)\ , \quad 1 < p < \infty\ , \tag{2.8}$$

and

$$\big(H^{s_0}_{p}(\mathbb{R}^n,E), H^{s_1}_{p}(\mathbb{R}^n,E)\big)_{\theta,q} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n,E)\ , \quad p < \infty\ . \tag{2.9}$$

Finally, using the convention $1 = 1/p + 1/p'$ we have

$$\big[H^s_p(\mathbb{R}^n,E)\big]' \doteq H^{-s}_{p'}(\mathbb{R}^n,E')\ , \quad s \in \mathbb{R}\ , \quad 1 < p < \infty\ ,$$

and

$$\big[B^s_{p,q}(\mathbb{R}^n,E)\big]' \doteq B^{-s}_{p',q'}(\mathbb{R}^n,E')\ , \quad s \in \mathbb{R}\ , \quad 1 \leq p,q < \infty\ ,$$

with respect to the duality pairing naturally induced by the L_p-duality pairing.

For $\Omega \subset \mathbb{R}^n$ open, we denote by $r_\Omega \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n,E), \mathcal{D}'(\Omega,E)\big)$ the restriction operator in the sense of distribution, and we put $\mathcal{F}(\Omega,E) := r_\Omega \mathcal{F}(\mathbb{R}^n,E)$ for $\mathcal{F} \in \big\{H^s_p, B^s_{p,q}\big\}$. Equipped with the quotient-space topology, these are Banach spaces.

For the remainder, let Ω be a bounded, open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Then it follows from [7, Theorem 4.1, Proposition 4.2] that, for

$$\mathcal{F} \in \big\{W^s_p\,,\, B^s_{p,q}\,,\, BUC^s\,;\, s \in \mathbb{R}\,,\, 1 \leq p < \infty\,,\, 1 \leq q \leq \infty\big\}\ ,$$

where $s \geq 0$ if $\mathcal{F} = BUC^s$, the operator $r_\Omega \in \mathcal{L}\big(\mathcal{F}(\mathbb{R}^n,E), \mathcal{F}(\Omega,E)\big)$ is a retraction. Moreover, there exists a universal coretraction e_Ω independent of p, q, and s. Interpolation entails that

$$r_\Omega \in \mathcal{L}\big(H^s_p(\mathbb{R}^n,E), H^s_p(\Omega,E)\big)\ , \quad s \in \mathbb{R}\ , \quad 1 < p < \infty\ ,$$

is a retraction as well. Therefore, statements (2.1)–(2.9) remain true if $\mathbb{R}^n$ is replaced by Ω. Given a Hilbert space F, one can prove, as in the scalar case, that, for $1 < p < \infty$, $1 \leq q < \infty$, and $-1/p' < s < 1/p$,

$$\big[H^s_p(\Omega,F)\big]' \doteq H^{-s}_{p'}(\Omega,F) \qquad \text{and} \qquad \big[B^s_{p,q}(\Omega,F)\big]' \doteq B^{-s}_{p',q'}(\Omega,F) \tag{2.10}$$

with respect to the duality pairing induced by the L_p-duality pairing.

By means of local coordinates and a partition of unity one then defines the Banach spaces $\mathcal{F}(\partial\Omega, E)$ for $\mathcal{F} \in \big\{H_p^s\,,\, B_{p,q}^s\big\}$ analogously to [38]. As in the scalar case it follows that $\partial_\nu^m : \big[u \mapsto \partial_\nu^m u = \frac{\partial^m}{\partial\nu^m} u\big]$ induces for each $m \in \mathbb{N}$ an operator

$$\partial_\nu^m \in \mathcal{L}\big(H_p^s(\Omega, E), B_{p,p}^{s-m-1/p}(\partial\Omega, E)\big) \cap \mathcal{L}\big(B_{p,q}^s(\Omega, E), B_{p,q}^{s-m-1/p}(\partial\Omega, E)\big)$$

provided $s > m + 1/p$, $1 < p < \infty$, and $1 \leq q \leq \infty$. We refrain from giving details and refer to [42, Section 6]. Hence, given

$$\mathcal{F} \in \big\{H_p^s\,,\, B_{p,q}^s\,;\, s \geq 0,\, 1 < p < \infty,\, 1 \leq q \leq \infty\big\}\ ,$$

it makes sense to define the Banach spaces

$$\mathcal{F}_{\mathcal{B}_m}(\Omega, E) := \begin{cases} \big\{u \in \mathcal{F}(\Omega, E)\,;\, \partial_\nu^m u = 0\big\}\ , & s > m + 1/p\ , \\ \mathcal{F}(\Omega, E)\ , & 0 \leq s < m + 1/p\ , \end{cases}$$

for $m \in \mathbb{N}$.

We conclude this section with a theorem on interpolation with boundary conditions. If the underlying space E is finite dimensional, these results are well-known and due to Grisvard [20, 21] and Seeley [36]. Our theorem extends these results to infinite-dimensional Hilbert spaces.

Theorem 2.1. *Suppose that E is a Hilbert space, and let $m \in \mathbb{N}$, $1 < p < \infty$, and $1 \leq q \leq \infty$. Given $\theta \in (0,1)$ and $s_1 > s_0 \geq 0$, put $s_\theta := (1-\theta)s_0 + \theta s_1$. Then, provided $s_1,\ s_\theta \neq m + 1/p$, it holds that*

$$\big[H_p^{s_0}(\Omega, E), H_{p,\mathcal{B}_m}^{s_1}(\Omega, E)\big]_\theta \doteq H_{p,\mathcal{B}_m}^{s_\theta}(\Omega, E)$$

and

$$\big(H_p^{s_0}(\Omega, E), H_{p,\mathcal{B}_m}^{s_1}(\Omega, E)\big)_{\theta,q} \doteq B_{p,q;\mathcal{B}_m}^{s_\theta}(\Omega, E)\ .$$

Proof. The argumentation follows the lines of the proofs of Guidetti [22, Theorem 2.3 and Theorem 2.7], where the scalar-valued analogue is shown in the case of Besov spaces. Basically, two main ingredients are required. First one has to ensure that the characteristic function $\chi_{\mathbb{R}_+^n}$ of the half space $\mathbb{R}_+^n$ is a multiplier for the space $H_p^s(\mathbb{R}^n, E)$ if $1 < p < \infty$ and $-1 + 1/p < s < 1/p$. This might be done by adapting the proof of the scalar-valued counterpart of Strichartz [37]. The basic observation for this result is that the first part of the statement of [37, 2.3. Third Characterization] remains valid for E-valued Bessel potential spaces provided E is Hilbertian. This is due to [35, Remark 6, Proposition 8]. The second requirement is the existence of an operator

$$Q_m \in \mathcal{L}\Big(\prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\partial\Omega, E), H_p^s(\Omega, E)\Big)\ , \quad s \in \mathbb{R}\ , \quad 1 < p < \infty\ ,$$

such that, for each $k \in \{0, \ldots, m\}$ with $s > k + 1/p$,

$$\partial_\nu^k Q_m(u^0, \ldots, u^m) = u^k\ , \quad (u^0, \ldots, u^m) \in \prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\partial\Omega, E)\ .$$

This result is obtained by reducing the problem to a full-space problem by means of local coordinates and using then similar arguments as in [2, Appendix B] and [39, Theorem 2.9.2/1]. A detailed proof of this theorem can be found in [42, Section 6]. □

3. The diffusion semigroup

If $E_1 \hookrightarrow E_0$ are densely injected Banach spaces, and if $A : E_1 \to E_0$ is linear, we mean by writing $A \in \mathcal{H}(E_1, E_0)$ that $-A$, considered as a linear operator in E_0 with domain E_1, generates an analytic semigroup on E_0.

For the remainder, Ω always denotes a bounded and smooth domain in $\mathbb{R}^n$. In order to simplify the notation, we put $\mathcal{F}[E] := \mathcal{F}(\Omega, E)$, where $\mathcal{F}(\Omega, E)$ is any space of functions defined on Ω with values in a Banach space E. Recall that $Y = (0, y_0]$. Given a function $d : \mathbb{R}^+ \to C(\bar{Y})$, we set

$$d(t, y) := d(t)(y)\ , \quad (t, y) \in \mathbb{R}^+ \times \bar{Y}\ .$$

Finally, we define, for a UMD space E, $H_{p,\mathcal{B}}^2[E] := \big\{u \in H_p^2[E]\,;\, \partial_\nu u = 0\big\}$. With this notation we can state the following fundamental theorem.

Theorem 3.1. *Let $\rho \in (0,1)$, and suppose that*

$$d \in C^\rho\big(\mathbb{R}^+, C(\bar{Y})\big) \quad \textit{with} \quad d(t, y) > 0\ , \quad (t, y) \in \mathbb{R}^+ \times \bar{Y}\ . \tag{3.1}$$

For $1 < p, \sigma < \infty$ put

$$A_p[d](t)u := -d(t, \cdot)\Delta u\ , \quad u \in H_{p,\mathcal{B}}^2[L_\sigma(Y)]\ , \quad t \geq 0\ . \tag{3.2}$$

Then it holds that

$$\big[t \mapsto A_p[d](t)\big] \in C^\rho\Big(\mathbb{R}^+, \mathcal{H}\big(H_{p,\mathcal{B}}^2[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)\Big)\ .$$

Proof. This follows from [15, Theorem 8.2] (see [42, Section 7.2]). □

Under the assumptions of Theorem 3.1, [4, II. Corollary 4.4.2] guarantees the existence of an evolution operator U_{A_p} of $A_p := A_p[d]$ on $L_p[L_\sigma(Y)]$. In the following, we collect some basic properties of U_{A_p}.

Lemma 3.2. *Suppose that d satisfies* (3.1). *Then, given $\sigma \in (1,\infty)$, it holds that*

$$U_{A_p}\big|_{L_q[L_\sigma(Y)]} = U_{A_q} \ , \quad 1 < p < q < \infty \ .$$

Proof. This is a consequence of the fact that

$$\big(\lambda + A_p(s)\big)^{-1}\big|_{L_q[L_\sigma(Y)]} = \big(\lambda + A_q(s)\big)^{-1} \ ,$$

for all $s \geq 0$ and $\lambda \in \mathbb{R}$ sufficiently large (cf. [42, Lemma 7.6]). □

The aim of the last part of this section is to prove that U_{A_p} is a positive operator. To this end, let us introduce some notation.

Given a vector space X ordered by a proper cone X^+ (that is, $x \leq y$ if and only if $y - x \in X^+$ with the convention that $y \geq x$ if and only if $x \leq y$) and any set M, the vector space X^M is given its pointwise order induced by the cone $\big(X^+\big)^M$. Thus $w \leq v$ for $w, v \in X^M$ if and only if $w(m) \leq v(m)\,,\ m \in M$. Let X be an ordered Banach space. Then $L_\sigma(Y)$ and $L_p[X]$ are ordered Banach spaces as well (with pointwise order almost everywhere), their cones being denoted by $L_\sigma^+(Y)$ and $L_p^+[X]$.

Given $\mathcal{F} \in \big\{BUC^\mu\,,\ H_p^\mu\,,\ B_{p,q}^\mu\,;\ \mu > 0\big\}$, the order of $\mathcal{F}[X]$ is defined by the cone $\mathcal{F}[X] \cap L_p^+[X]$. By $C_c^+(Y)$ we denote the positive cone of $C_c(Y)$.

$T \in \mathcal{L}(X)$ is said to be *positive* if $T(X^+) \subset X^+$. A closed linear operator A in X is *resolvent positive* provided there exists $\lambda_0 \geq 0$ such that $[\lambda_0, \infty)$ belongs to the resolvent set $\varrho(-A)$ of $-A$, and $(\lambda + A)^{-1} \in \mathcal{L}(X)$ is positive for $\lambda \geq \lambda_0$.

Lemma 3.3. *For $1 < p, \sigma < \infty$, the tensor product $BUC^\infty(\Omega)^+ \otimes C_c^+(Y)$ is dense in $L_p^+[L_\sigma(Y)]$ and $BUC^\infty(\Omega) \otimes C_c(Y)$ is dense in $H_p^2[L_\sigma(Y)]$.*

Proof. Due to the fact that the trivial extension $\tilde{v}$ of $v \in L_p^+[L_\sigma(Y)]$ belongs to $L_p^+\big(\mathbb{R}^n, L_\sigma(Y)\big)$, the first part of the statement follows analogously to [6, Lemma 6.1]. The second part is obtained similarly by extending elements of $H_p^2[L_\sigma(Y)]$ to $H_p^2\big(\mathbb{R}^n, L_\sigma(Y)\big)$ by means of Section 2. □

Theorem 3.4. *Suppose that d satisfies* (3.1). *Then, for $1 < p, \sigma < \infty$, the evolution operator U_{A_p} of $A_p = A_p[d]$ is a positive operator on $L_p[L_\sigma(Y)]$.*

Proof. In view of [4, II. Theorem 6.4.1 and II. Theorem 6.4.2] it suffices to prove that, for fixed $s \geq 0$, the closed linear operator $B_p := A_p(s)$ in $L_p[L_\sigma(Y)]$ is resolvent positive.

(i) Assume $\sigma \geq p > n$. From (the proof of) [1, Theorem 6.1] follows the existence of $\lambda_0 \in \mathbb{R}$ such that, for each $y \in Y$ and $\lambda \geq \lambda_0$, we have $w \geq 0$

whenever $w \in H^2_{p,\mathcal{B}}(\Omega)$ satisfies $\big(\lambda - d(s,y)\Delta\big)w \geq 0$. Due to

$$B_p \in \mathcal{H}\big(H^2_{p,\mathcal{B}}[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)$$

there is $\omega_p > 0$ such that $[\omega_p, \infty) \subset \varrho(-B_p)$. Fix $\lambda \geq \max\{\omega_p, \lambda_0\} =: \lambda(p)$, and let v belong to $BUC^\infty(\Omega)^+ \otimes C_c^+(Y)$ so that

$$w := (\lambda + B_p)^{-1} v \in H^2_{p,\mathcal{B}}[L_\sigma(Y)] \hookrightarrow H^2_{p,\mathcal{B}}[L_p(Y)] \ , \tag{3.3}$$

whence

$$(\lambda + B_p)w = v \geq 0 \quad \text{in} \quad L_p[L_\sigma(Y)] \hookrightarrow L_p[L_p(Y)] \ . \tag{3.4}$$

Recalling $L_p[L_p(Y)] = L_p\big(Y, L_p(\Omega)\big)$ and the fact that $BUC^\infty(\Omega) \otimes C_c(Y)$ is dense in $H^2_p[L_p(Y)]$, we derive from (3.3) and (3.4) that $w(\cdot, y) := w(\cdot)(y)$ belongs to $H^2_{p,\mathcal{B}}(\Omega)$ for almost every $y \in Y$ and satisfies

$$\big(\lambda - d(s,y)\Delta\big)w(\cdot,y) = v(\cdot,y) \geq 0 \ .$$

Thus $w(x,y) \geq 0$ for each $x \in \Omega$ and almost every $y \in Y$. We conclude

$$(\lambda + B_p)^{-1} v \geq 0 \ , \quad v \in BUC^\infty(\Omega)^+ \otimes C_c^+(Y) \ , \quad \lambda \geq \lambda(p) \ . \tag{3.5}$$

Next use Lemma 3.3 and the closedness of the positive cone $L_p^+[L_\sigma(Y)]$ in $L_p[L_\sigma(Y)]$ to deduce that (3.5) remains valid for $v \in L_p^+[L_\sigma(Y)]$.

(ii) Assume that $1 < p, \sigma < \infty$ are arbitrary. Choose $\tau \geq \sigma$ and $q \geq p$ with $\tau \geq q > n$. Fix $\omega_p > 0$ such that $[\omega_p, \infty)$ belongs to the resolvent set of $-B_p$ and put $\omega := \max\{\omega_p, \lambda(q)\}$, where $\lambda(q)$ is given as in (i). According to Lemma 3.3, $L_q^+[L_\tau(Y)]$ is dense in $L_p^+[L_\sigma(Y)]$. Therefore, since $(\lambda + B_p)^{-1}$ is for any $\lambda \geq \omega$ a bounded and linear operator on $L_p[L_\sigma(Y)]$ satisfying (see the proof of Lemma 3.2)

$$(\lambda + B_p)^{-1}\big|_{L_q[L_\tau(Y)]} = (\lambda + B_q)^{-1} \in \mathcal{L}\big(L_q[L_\tau(Y)]\big) \ ,$$

the assertion follows from (i). $\square$

For $1 < p < \infty$ and $1 \leq q < \infty$ we denote by $\big\{S^\mu_{p,q}[L_2(Y)]\,;\, \mu \neq 0\big\}$ either the scale $\big\{B^\mu_{p,q}[L_2(Y)]\,;\, \mu \neq 0\big\}$ or the scale $\big\{H^\mu_p[L_2(Y)]\,;\, \mu \neq 0\big\}$, and we set $S^0_{p,q}[L_2(Y)] := L_p[L_2(Y)]$. Moreover, we put

$$S^\mu_{p,q;\mathcal{B}}[L_2(Y)] := \begin{cases} \big\{u \in S^\mu_{p,q}[L_2(Y)]; \partial_\nu u = 0\big\}, & \mu > 1 + \frac{1}{p}, \\ S^\mu_{p,q}[L_2(Y)], & 0 < \mu < 1 + \frac{1}{p}, \end{cases} \tag{3.6}$$

and $S^{\mu,+}_{p,q;\mathcal{B}}[L_2(Y)] := S^\mu_{p,q;\mathcal{B}}[L_2(Y)] \cap L_p^+[L_2(Y)]$ for $\mu > 0$.

Corollary 3.5. *Let $1 < p < \infty$ and $1 \leq q < \infty$, and suppose that $0 < \mu \leq \eta < 2$ with $\mu, \eta \neq 1 + 1/p$. Then $S^{\eta,+}_{p,q;\mathcal{B}}[L_2(Y)]$ is dense in $S^{\mu,+}_{p,q;\mathcal{B}}[L_2(Y)]$.*

Proof. Taking into account Theorem 2.1 and (the proof of) Theorem 3.4, the statement is due to [4, V. Proposition 2.7.1]. $\square$

We conclude this section with the following lemma characterizing the dual operator $\big(A_p[d]\big)'$ of $A_p[d]$ in the case $\sigma := 2$.

Lemma 3.6. *Suppose that $d \in C\big(\bar{Y}, (0,\infty)\big)$ and put, for $1 < p < \infty$,*

$$A_p := A_p[d] \in \mathcal{H}\big(H^2_{p,\mathcal{B}}[L_2(Y)], L_p[L_2(Y)]\big) \ .$$

Then it holds that $(A_p)' = A_{p'}$ with respect to the $L_p[L_2(Y)]$-$L_{p'}[L_2(Y)]$-duality pairing.

Proof. The assertion is a consequence of $\big(L_p[L_2(Y)]\big)' = L_{p'}[L_2(Y)]$ and Gauss' theorem. $\square$

4. The reaction terms

Recall that $Y = (0, y_0]$ denotes the admissible range of droplet masses. For abbreviation we put $E := L_2(Y)$, and we assume throughout that $1 < p < \infty$ and $1 \le q < \infty$ are fixed.

Let $E_0, \dots, E_m$ be Banach spaces. The Banach space $\mathcal{L}(E_1, \dots, E_m; E_0)$ consists of all continuous m-linear maps from $E_1 \times \cdots \times E_m$ into E_0. They are called *multiplications*. Sometimes we simply write

$$(e_1, \dots, e_m) \mapsto e_1 \bullet \cdots \bullet e_m$$

for a multiplication. Given a multiplication and $u_j \in (E_j)^{\Omega}$, $1 \le j \le m$, we define $u_1 \bullet \cdots \bullet u_m \in (E_0)^{\Omega}$ by

$$u_1 \bullet \cdots \bullet u_m(x) := u_1(x) \bullet \cdots \bullet u_m(x) \ , \quad x \in \Omega \ . \tag{4.1}$$

For any Banach spaces $F_j[E_j]$ of E_j-valued functions defined on Ω we write

$$F_1[E_1] \bullet \cdots \bullet F_m[E_m] \hookrightarrow F_0[E_0] \ ,$$

provided that (4.1) defines an element of $\mathcal{L}\big(F_1[E_1], \dots, F_m[E_m]; F_0[E_0]\big)$.

Lemma 4.1. (*a*) *Let $E_1 \times \cdots \times E_m \to E_0$, $(e_1, \dots, e_m) \mapsto e_1 \bullet \cdots \bullet e_m$, be a multiplication with $m \ge 3$. If*

$$0 < \tau < \min\{r, n/p\} \quad \textit{and} \quad \tau + n/p < 2\sigma \ , \tag{4.2}$$

then

$$BUC^r[E_1] \bullet \cdots \bullet BUC^r[E_{m-2}] \bullet B^{\sigma}_{p,q}[E_{m-1}] \bullet B^{\sigma}_{p,q}[E_m] \hookrightarrow B^{\tau}_{p,q}[E_0] \ .$$

(*b*) *Suppose that $E_1 \times E_2 \to E_0$, $(e_1, e_2) \mapsto e_1 \bullet e_2$, is a multiplication. If $0 < \sigma < r$, then*

$$BUC^r[E_1] \bullet B^{\sigma}_{p,q}[E_2] \hookrightarrow B^{\sigma}_{p,q}[E_0] \ .$$

Proof. By means of the retraction r_Ω and a corresponding coretraction e_Ω (see Section 2) we may assume $\Omega = \mathbb{R}^n$. Then the assertions are consequences of

$$B^r_{\infty,\infty}(\mathbb{R}^n, E_j) \doteq BUC^r(\mathbb{R}^n, E_j) \ , \quad r \in \mathbb{R}^+ \setminus \mathbb{N} \ ,$$

and [3, Theorem 4.1 and Remark 4.2(b)], if one observes that the results in [3] remain valid for arbitrary, not necessarily finite-dimensional Banach spaces (see [5, 8]). $\square$

The space $C_b^{1-}(E_1, E_0)$ consists of all maps from E_1 into E_0 which are uniformly Lipschitz continuous on bounded subsets of E_1. Endowed with the family of seminorms

$$p_B := \Big[u \mapsto \sup_{e \in B} \|u(e)\|_{E_0} + \sup_{\substack{e,e' \in B \\ e \neq e'}} \frac{\|u(e) - u(e')\|_{E_0}}{\|e - e'\|_{E_1}} \Big] \ ,$$

where B runs through the family of all bounded subsets of E_1, $C_b^{1-}(E_1, E_0)$ is a locally convex space. Then $C^\rho\big(\mathbb{R}^+, C_b^{1-}(E_1, E_0)\big)$ for $\rho \in (0,1)$ is a locally convex space as well, with topology induced by the family of seminorms

$$u \mapsto \max_{0 \le t \le T} p_B\big(u(t)\big) + \sup_{0 \le s < t \le T} \frac{p_B\big(u(t) - u(s)\big)}{|t-s|^\rho} \ ,$$

with $T > 0$ and $B \subset E_1$ bounded.

We set $F_{\mathsf{b}} := L_\infty(Y, E)$ and use the notation

$$\gamma(y, y') := \gamma(y)(y') \ , \quad y, y' \in Y \ , \quad \gamma \in F_{\mathsf{b}} \ .$$

Given $\gamma \in F_{\mathsf{b}}$, we define

$$l_{\mathsf{b}}(\gamma)[u](y) := \int_y^{y_0} \gamma(y', y) u(y') \, \mathrm{d}y' - u(y) \int_0^y \frac{y'}{y} \gamma(y, y') \, \mathrm{d}y'$$

for $u \in E$ and almost every $y \in Y$. Since Y is a bounded interval, it is easily seen that $\big[(\gamma, u) \mapsto l_{\mathsf{b}}(\gamma)[u]\big] \in \mathcal{L}(F_{\mathsf{b}}, E; E)$, and hence, putting

$$l_{\mathsf{b}}(\gamma)[u](x) := l_{\mathsf{b}}\big(\gamma(x)\big)[u(x)] \ , \quad x \in \Omega \ ,$$

for $(\gamma, u) : \Omega \to F_{\mathsf{b}} \times E$, Lemma 4.1(b) yields a multiplication

$$\big[(\gamma, u) \mapsto l_{\mathsf{b}}(\gamma)[u]\big] \in \mathcal{L}\big(BUC^r[F_{\mathsf{b}}], B^\sigma_{p,q}[E]; B^\sigma_{p,q}[E]\big) \ ,$$

provided $0 < \sigma < r$. If $\gamma \in C^\rho\big(\mathbb{R}^+, BUC^r[F_{\mathsf{b}}]\big)$ is fixed, where $\rho \in (0,1)$, we define

$$l_{\mathsf{b}}(t, x, y, u) := l_{\mathsf{b}}\big(\gamma(t)(x)\big)[u](y) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad \text{a.e. } y \in Y \ , \quad u \in E \ .$$

Denoting then by $L_{\mathsf{b}}(t,\cdot)$ the Nemytskii operator induced by $l_{\mathsf{b}}(t,\cdot,\cdot,\cdot)$, that is,

$$L_{\mathsf{b}}(t,u)(x) := l_{\mathsf{b}}\big(t,x,\cdot,u(x)\big)\ , \quad (t,x)\in\mathbb{R}^+\times\Omega\ , \quad u\in E^{\Omega}\ ,$$

we deduce

$$\big[t\mapsto L_{\mathsf{b}}(t,\cdot)\big]\in C^{\rho}\big(\mathbb{R}^+, C_b^{1-}\big(B^{\sigma}_{p,q}[E], B^{\sigma}_{p,q}[E]\big)\big)\ , \quad 0<\sigma<r\ .$$

Next, let F_{c} be the closed linear subspace of $L_\infty(Y\times Y)$ which consists of all R satisfying

$$R(y,y') = R(y',y)\ , \quad \text{a.e. } y,y'\in Y\ .$$

Defining

$$l^1_{\mathsf{c}}(K,P)[u,v](y) := \frac{1}{2}\int_0^y K(y',y-y')P(y',y-y')u(y')v(y-y')\ \mathrm{d}y'$$

for $K,P\in F_{\mathsf{c}}$, $u,v\in E$, and almost every $y\in Y$, we obtain a multiplication

$$\big[(K,P,u,v)\mapsto l^1_{\mathsf{c}}(K,P)[u,v]\big]\in\mathcal{L}(F_{\mathsf{c}},F_{\mathsf{c}},E,E;E)\ .$$

Similarly, the definitions of

$$\begin{aligned} l^2_{\mathsf{c}}(\beta_{\mathsf{c}},K,Q)[u,v](y) := \frac{1}{2}\int_y^{y_0}\int_0^{y'} & K(y'',y'-y'')Q(y'',y'-y'') \\ & \times \beta_{\mathsf{c}}(y',y)u(y'')v(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \end{aligned}$$

and

$$l^3_{\mathsf{c}}(K,R)[u,v](y) := u(y)\int_0^{y_0-y} K(y,y')R(y,y')v(y')\ \mathrm{d}y',$$

for $\beta_{\mathsf{c}}\in F_{\mathsf{b}}$, $K,Q,R\in F_{\mathsf{c}}$, $u,v\in E$, and almost every $y\in Y$, yield multiplications

$$\big[(\beta_{\mathsf{c}},K,Q,u,v)\mapsto l^2_{\mathsf{c}}(\beta_{\mathsf{c}},K,Q)[u,v]\big]\in\mathcal{L}(F_{\mathsf{b}},F_{\mathsf{c}},F_{\mathsf{c}},E,E;E)$$

and

$$\big[(K,R,u,v)\mapsto l^3_{\mathsf{c}}(K,R)[u,v]\big]\in\mathcal{L}(F_{\mathsf{c}},F_{\mathsf{c}},E,E;E)\ .$$

Furthermore, for $\beta_{\mathsf{s}}\in F_{\mathsf{s}} := L_\infty\big((y_0,2y_0],E\big)$, we write

$$\beta_{\mathsf{s}}(y,y') := \beta_{\mathsf{s}}(y)(y')\ , \quad y\in(y_0,2y_0]\ , \quad y'\in Y\ .$$

We then put

$$\begin{aligned} l^1_{\mathsf{s}}(\beta_{\mathsf{s}},K)[u,v](y) := \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} & K(y'',y'-y'')\beta_{\mathsf{s}}(y',y) \\ & \times u(y'')v(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \end{aligned}$$

and

$$l_{\mathsf{s}}^2(K)[u,v](y) := u(y)\int_{y_0-y}^{y_0} K(y,y')v(y')\,\mathrm{d}y'$$

for $\beta_{\mathsf{s}} \in F_{\mathsf{s}}$, $K \in F_{\mathsf{c}}$, $u,v \in E$, and almost every $y \in Y$. The mappings l_{s}^1 and l_{s}^2 have the property

$$\big[(\beta_{\mathsf{s}},K,u,v) \mapsto l_{\mathsf{s}}^1(\beta_{\mathsf{s}},K)[u,v]\big] \in \mathcal{L}(F_{\mathsf{s}},F_{\mathsf{c}},E,E;E)$$

and

$$\big[(K,u,v) \mapsto l_{\mathsf{s}}^2(K)[u,v]\big] \in \mathcal{L}(F_{\mathsf{c}},E,E;E)\ .$$

Suppose now that

$$(\beta_{\mathsf{c}},\beta_{\mathsf{s}},K,P,Q) \in C^\rho\big(\mathbb{R}^+,BUC^r[F_{\mathsf{b}}\times F_{\mathsf{s}}\times F_{\mathsf{c}}\times F_{\mathsf{c}}\times F_{\mathsf{c}}]\big)$$

is fixed, where $r>0$ and $\rho\in(0,1)$. We set

$$\begin{aligned}
l_{\mathsf{c}}(t,x,y,u,v) &:= l_{\mathsf{c}}^1(t,x,y,u,v) + l_{\mathsf{c}}^2(t,x,y,u,v) - l_{\mathsf{c}}^3(t,x,y,u,v)\\
&:= l_{\mathsf{c}}^1\big(K(t)(x),P(t)(x)\big)[u,v](y)\\
&\qquad + l_{\mathsf{c}}^2\big(\beta_{\mathsf{c}}(t)(x),K(t)(x),Q(t)(x)\big)[u,v](y)\\
&\qquad - l_{\mathsf{c}}^3\big(K(t)(x),P(t)(x)+Q(t)(x)\big)[u,v](y)
\end{aligned}$$

as well as

$$\begin{aligned}
l_{\mathsf{s}}(t,x,y,u,v) &:= l_{\mathsf{s}}^1(t,x,y,u,v) - l_{\mathsf{s}}^2(t,x,y,u,v)\\
&:= l_{\mathsf{s}}^1\big(\beta_{\mathsf{s}}(t)(x),K(t)(x)\big)[u,v](y) - l_{\mathsf{s}}^2\big(K(t)(x)\big)[u,v](y)
\end{aligned}$$

for $(t,x)\in\mathbb{R}^+\times\Omega$, $u,v\in E$, and almost every $y\in Y$. Moreover, we denote by $L_h^j(t,\cdot,\cdot)$ for $(j,h)\in\big\{(1,c),(2,c),(3,c),(1,s),(2,s)\big\}$ the Nemytskii operator induced by $l_h^j(t,\cdot,\cdot,\cdot,\cdot)$; that is,

$$L_h^j(t,u,v)(x) := l_h^j\big(t,x,\cdot,u(x),v(x)\big)\ ,\quad (t,x)\in\mathbb{R}^+\times\Omega\ ,\quad u,v\in E^\Omega\ ,$$

and we put

$$L_{\mathsf{c}}(t,u) := L_{\mathsf{c}}^1(t,u,u) + L_{\mathsf{c}}^2(t,u,u) - L_{\mathsf{c}}^3(t,u,u)\ ,\quad t\in\mathbb{R}^+\ ,\quad u\in E^\Omega\ ,$$

and

$$L_{\mathsf{s}}(t,u) := L_{\mathsf{s}}^1(t,u,u) - L_{\mathsf{s}}^2(t,u,u)\ ,\quad t\in\mathbb{R}^+\ ,\quad u\in E^\Omega\ .$$

Therefore, by virtue of Lemma 4.1, these operators satisfy

$$\big[t\mapsto L_h(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+,C_b^{1-}\big(B_{p,q}^\sigma[E],B_{p,q}^\tau[E]\big)\big)\ ,\quad h\in\{c,s\}\ ,$$

provided (4.2) holds. Finally, we set

$$L(t,\cdot) := L_{\mathsf{b}}(t,\cdot) + L_{\mathsf{c}}(t,\cdot) + L_{\mathsf{s}}(t,\cdot)\ ,\quad t\in\mathbb{R}^+\ , \tag{4.3}$$

and

$$\mathbb{F} := F_{\mathsf{b}} \times F_{\mathsf{b}} \times F_{\mathsf{s}} \times F_{\mathsf{c}} \times F_{\mathsf{c}} \times F_{\mathsf{c}} \ . \tag{4.4}$$

We summarize the observations above in the following proposition.

Proposition 4.2. *Assume* $1 < p < \infty$ *and* $1 \le q < \infty$, *and let*

$$0 < \tau < \min\{r, n/p\} \quad \textit{with} \quad \tau + n/p < 2\sigma \ .$$

Suppose that

$$\Big[t \mapsto \big(\gamma(t), \beta_{\mathsf{c}}(t), \beta_{\mathsf{s}}(t), K(t), P(t), Q(t)\big)\Big] \in C^\rho\big(\mathbb{R}^+, BUC^r[\mathbb{F}]\big)$$

for some $\rho \in (0,1)$. *Then it holds that*

$$\big[t \mapsto L(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+, C_b^{1-}\big(B^\sigma_{p,q}[E], B^\tau_{p,q}[E]\big)\big) \ .$$

For the sake of readability we will use in the sequel the notation

$$a(t,x;\cdot,\cdot) := a(t)(x)(\cdot,\cdot) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad a \in \big\{\gamma, \beta_{\mathsf{c}}, \beta_{\mathsf{s}}, K, P, Q\big\} \ .$$

5. Well-posedness

Now we are in a position to prove that problem $(*)$ is well-posed. To this end, let us rewrite the equations in $(*)$ as a Banach-space-valued Cauchy problem of the form

$$\begin{aligned} \dot u + A(t)u &= L(t,u) \ , \quad t > 0 \ , \\ u(0) &= u^0 \ , \end{aligned} \tag{$(CP)_{u^0}$}$$

where $L(t,\cdot)$ and $A(t) := A_p[d](t) \in \mathcal{H}\big(H^2_{p,\mathcal{B}}[E], L_p[E]\big)$ are given by (4.3) and (3.2), respectively, and where $E = L_2(Y)$. Recall that we defined the scale $\big\{S^\mu_{p,q;\mathcal{B}}[E]\,;\,\mu > 0\big\}$ in (3.6), and that $\mathbb{F}$ is given by (4.4).

In the sequel, we put $\dot J := J \setminus \{0\}$ for any interval $J \subset \mathbb{R}$.

5.1. **Existence and uniqueness.**

The following theorem guarantees existence and uniqueness of maximal solutions to problem $(CP)_{u^0}$ in $L_p[E]$.

Theorem 5.1. *Let* $r > 0$ *and* $\rho \in (0,1)$, *and suppose that*

$$\big[t \mapsto \big(\gamma(t), \beta_{\mathsf{c}}(t), \beta_{\mathsf{s}}(t), K(t), P(t), Q(t)\big)\big] \in C^\rho\big(\mathbb{R}^+, BUC^r[\mathbb{F}]\big)$$

and

$$d \in C^\rho\big(\mathbb{R}^+, C(\bar Y)\big) \quad \textit{with} \quad d(t,y) > 0 \ , \quad (t,y) \in \mathbb{R}^+ \times \bar Y \ .$$

Also suppose $1 < p < \infty$ *and* $1 \le q < \infty$, *and let* $n < 4p$ *and* $\mu \in (n/2p\,,\,2)$ *with* $\mu \ne 1 + 1/p$. *Then, given any* $u^0 \in S^\mu_{p,q;\mathcal{B}}[E]$, *problem* $(CP)_{u^0}$ *possesses a unique maximal solution* $u := u(\cdot;u^0)$ *satisfying*

$$u \in C\big(J(u^0), S^\mu_{p,q;\mathcal{B}}[E]\big) \cap C^1\big(\dot J(u^0), L_p[E]\big) \cap C\big(\dot J(u^0), H^2_{p,\mathcal{B}}[E]\big) \ .$$

The maximal interval of existence $J(u^0)$ is open in $\mathbb{R}^+$. If

$$\sup_{t\in J(u^0)\cap[0,T]} \|u(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]} < \infty\ ,\quad T>0\ , \tag{5.1}$$

then $J(u^0)=\mathbb{R}^+$. Moreover, the solution $u(\cdot;u^0)$ depends continuously on the initial value u^0 in the following sense: For each $T\in \dot J(u^0)$ there exists a neighborhood U of u^0 in $S^\mu_{p,q;\mathcal{B}}[E]$ such that $J(v^0)\supset[0,T]$ for $v^0\in U$ and, as $v^0\to u^0$ in U,

$$u(\cdot;v^0)\to u(\cdot;u^0)\quad in\quad C\big([0,T],S^\mu_{p,q;\mathcal{B}}[E]\big)\ .$$

Proof. Set $(\mathbb{E}_0,\mathbb{E}_1):=\big(L_p[E],H^2_{p,\mathcal{B}}[E]\big)$, and put for $\theta\in(0,1)$

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,q} & \text{if}\quad \big\{S^\mu_{p,q;\mathcal{B}}[E]\,;\,\mu>0\big\}=\big\{B^\mu_{p,q;\mathcal{B}}[E]\,;\,\mu>0\big\}\ ,\\ [\cdot,\cdot]_\theta & \text{if}\quad \big\{S^\mu_{p,q;\mathcal{B}}[E]\,;\,\mu>0\big\}=\big\{H^\mu_{p,\mathcal{B}}[E]\,;\,\mu>0\big\}\ .\end{cases}$$

Clearly, $\mathbb{E}_1\overset{d}{\hookrightarrow}\mathbb{E}_0$. Furthermore, Theorem 2.1 entails

$$\mathbb{E}_\theta:=(\mathbb{E}_0,\mathbb{E}_1)_\theta\doteq S^{2\theta}_{p,q;\mathcal{B}}[E]\ ,\quad 2\theta\in(0,2)\setminus\{1+1/p\}\ .$$

Fix $\sigma\in(n/2p\,,\,\mu)\setminus\{1+1/p\}$ and $\tau\in\big(0,\min\{2\sigma-n/p,1+1/p,r,n/p\}\big)$. Then, due to Proposition 4.2, we have

$$\big[t\mapsto L(t,\cdot)\big]\in C^\rho\big(\mathbb{R}^+,C^{1-}_b\big(B^\sigma_{p,q;\mathcal{B}}[E],B^\tau_{p,q;\mathcal{B}}[E]\big)\big)\ .$$

Choose $\varepsilon>0$ small such that

$$0<\vartheta_0:=\tau/2-\varepsilon<\vartheta_1:=\sigma/2+\varepsilon<\vartheta_2:=\mu/2<1\ .$$

Since, according to Section 2,

$$H^\xi_p[E]\hookrightarrow B^\eta_{p,q}[E]\hookrightarrow H^\zeta_p[E]\ ,\quad \zeta<\eta<\xi\ ,$$

Theorem 3.1 yields

$$\big[t\mapsto\big(A(t),L(t,\cdot)\big)\big]\in C^\rho\big(\mathbb{R}^+,\mathcal{H}(\mathbb{E}_1,\mathbb{E}_0)\times C^{1-}_b(\mathbb{E}_{\vartheta_1},\mathbb{E}_{\vartheta_0})\big)\ .$$

Owing to $u^0\in\mathbb{E}_{\vartheta_2}$, the assertion is now a consequence of [6, Theorem 5.1]. □

Remark 5.2. The solution $u(\cdot;u^0)$ has, in addition, the regularity

$$u(\cdot;u^0)\in C^{(\mu-\eta)/2}\big(J(u^0),S^\eta_{p,q;\mathcal{B}}[E]\big)\ ,\quad \eta\in(0,\mu]\setminus\{1+1/p\}\ ,$$

according to [4, II. Theorem 5.3.1].

Henceforth, in order to simplify the notation, we will write

$$u(t,x,y):=u(t;u^0)(x)(y)\ ,\quad (t,x,y)\in J(u^0)\times\Omega\times Y\ ,$$

for the solution $u=u(\cdot;u^0)$ to problem $(CP)_{u^0}$, and for convenience we will sometimes suppress any of the variables t, x, and y in a given formula.

5.2. **Conservation of mass.** The purpose of the next theorem is to provide sufficient conditions for mass conservation. Suppose that both scattering and collisional breakage are mass-preserving. More precisely, assume that, for each $(t,x) \in \mathbb{R}^+ \times \Omega$,

$$\int_0^{y_0} y''\beta_{\mathsf{s}}(t,x;y+y',y'')\ \mathrm{d}y'' = y+y' \ , \quad \text{a.e. } y_0 < y+y' \le 2y_0 \ , \tag{5.2}$$

and, for almost every $0 < y+y' \le y_0$,

$$Q(t,x;y,y')\Big[\int_0^{y+y'} y''\beta_{\mathsf{c}}(t,x;y+y',y'')\ \mathrm{d}y'' - y - y'\Big] = 0 \ . \tag{5.3}$$

Observe that assuming (5.3) and $\beta_{\mathsf{c}} \in C(\mathbb{R}^+, BUC[F_{\mathsf{b}}])$ restricts the physical scope of applications since in combination they imply that collisions of small droplets cannot result in a shattering. Indeed, Hölder's inequality and (5.3) entail that, for any $T > 0$,

$$\begin{aligned} Q(t,x;y,y')(y+y') &\le Q(t,x;y,y') \\ &\qquad \times \Big(\int_0^{y+y'} z^2\ \mathrm{d}z \int_0^{y+y'} \big|\beta_{\mathsf{c}}(t,x;y+y',y'')\big|^2\ \mathrm{d}y''\Big)^{1/2} \\ &\le c(T)\, Q(t,x;y,y')\ \big(y+y'\big)^{3/2} \end{aligned}$$

so that there exists $\mathsf{y}(T) \in Y$ with

$$Q(t,x;y,y') = 0 \ , \quad \text{a.e. } 0 < y+y' < \mathsf{y}(T) \ , \quad (t,x) \in [0,T]\times\Omega \ . \tag{5.4}$$

We point out that this restriction is of purely mathematical nature and not substantiated in the physical model.

Theorem 5.3. *Presuppose the hypotheses of Theorem* 5.1, *and let in addition* (5.2) *and* (5.3) *be valid. Then, for each* $u^0 \in S^{\mu}_{p,q;\mathcal{B}}[E]$, *the solution* $u(\cdot;u^0)$ *conserves the total mass; that is,*

$$\int_\Omega\int_Y yu(t;u^0)\ \mathrm{d}y\,\mathrm{d}x = \int_\Omega\int_Y yu^0\ \mathrm{d}y\,\mathrm{d}x \ , \quad t \in J(u^0) \ .$$

Proof. Since Ω and Y are bounded, Theorem 5.1 entails

$$\mathcal{M} := \Big[t \mapsto \int_\Omega\int_Y yu(t)\ \mathrm{d}y\,\mathrm{d}x\Big] \in C^1\big(\dot J(u^0)\big)\cap C\big(J(u^0)\big) \ .$$

Approximating $u(t) \in H^2_p[E]$ by elements of $BUC^\infty(\Omega)\otimes C_c(Y)$, we obtain the equality

$$\int_\Omega\int_Y yA(t)u(t)\ \mathrm{d}y\,\mathrm{d}x = -\int_{\partial\Omega}\int_Y yd(t,y)\partial_\nu u(t)\ \mathrm{d}y\,\mathrm{d}\sigma = 0 \ , \quad t \in \dot J(u^0) \ .$$

Proposition 4.2 yields $L\big(t,u(t)\big)\in L_1[L_1(Y)]$, $t\in\dot J(u^0)$, whence

$$\int_\Omega\int_Y yL\big(t,u(t)\big)\,\mathrm{d}y\,\mathrm{d}x=0\ ,\quad t\in\dot J(u^0)\ ,$$

by (5.2) and (5.3) (see [41, Lemma 2.6]). Consequently, $\dot{\mathcal M}(t)=0$ for $t\in\dot J(u^0)$. □

5.3. **Positivity.** Recall that the space $\mathbb F$, defined by (4.4), is an ordered Banach space with positive cone

$$\mathbb F^+:=F_{\mathsf b}^+\times F_{\mathsf b}^+\times F_{\mathsf s}^+\times F_{\mathsf c}^+\times F_{\mathsf c}^+\times F_{\mathsf c}^+\ ,$$

since the spaces $F_{\mathsf b}$, $F_{\mathsf s}$, and $F_{\mathsf c}$ are themselves ordered Banach spaces.

Theorem 5.4. *In addition to the assumptions of Theorem* 5.1 *suppose that*

$$\big(\gamma(t),\beta_{\mathsf c}(t),\beta_{\mathsf s}(t),K(t),P(t),Q(t)\big)\in BUC^r[\mathbb F]^+\ ,\quad t\ge 0\ . \tag{5.5}$$

Then $u^0\in S^{\mu,+}_{p,q;\mathcal B}[E]$ *implies* $u(t;u^0)\in S^{\mu,+}_{p,q;\mathcal B}[E]$, $t\in J(u^0)$.

Proof. (i) Assume $n<2p$ and $\mu\in(n/p\,,2)\setminus\{1+1/p\}$. According to Section 2 we have in this case, for $u^0\in S^{\mu}_{p,q;\mathcal B}[E]$,

$$u\in C\big(J(u^0),S^{\mu}_{p,q;\mathcal B}[E]\big)\hookrightarrow C\big(J(u^0),BUC[E]\big)\ .$$

Thus, for fixed $T_0\in\dot J(u^0)$, we can choose $\omega:=\omega(T_0)>0$ such that

$$\Big|\int_0^{y_0-y}K(t,x;y,y')\big[P(t,x;y,y')+Q(t,x;y,y')\big]u(t,x,y')\ \mathrm{d}y'\Big|\le\omega/2$$

and

$$\Big|\int_{y_0-y}^{y_0}K(t,x;y,y')u(t,x,y')\ \mathrm{d}y'\Big|+\Big|\int_0^y\frac{y'}{y}\gamma(t,x;y,y')\ \mathrm{d}y'\Big|\le\omega/2$$

for $(t,x)\in[0,T_0]\times\Omega$ and almost every $y\in Y$. Putting, for $0\le t\le T_0$ and $v\in C[E]$,

$$\begin{aligned}G(t,v):=&L_{\mathsf b}(t,v)+L^1_{\mathsf c}(t,v,v)+L^2_{\mathsf c}(t,v,v)+L^1_{\mathsf s}(t,v,v)\\&-L^3_{\mathsf c}\big(t,v,u(t)\big)-L^2_{\mathsf s}\big(t,v,u(t)\big)+\omega v\ ,\end{aligned}$$

where the operators $L^j_h(t,\cdot,\cdot)$ are defined as in Section 4, it follows that

$$G\big(t,v(t)\big)\ge 0\ ,\quad 0\le t\le T\le T_0\ ,\quad v\in C\big([0,T],C^+[E]\big)\ . \tag{5.6}$$

Moreover, since $G\big(t,u(t)\big)=L\big(t,u(t)\big)+\omega u(t)$, $0\le t\le T_0$, we see that u is a solution to

$$\dot v+B(t)v=G(t,v)\ ,\quad 0<t\le T_0\ ,\qquad v(0)=u^0$$

in $L_p[E]$, where $B := \omega + A \in C^\rho\big(\mathbb{R}^+, \mathcal{H}\big(H^2_{p,\mathcal{B}}[E], L_p[E]\big)\big)$. Denote by U_B the evolution operator of B. Choosing $M > 0$ and $T \in (0, T_0]$ appropriately, one easily proves on the basis of (a slight modification of) Proposition 4.2 and [4, II. Lemma 5.1.3] that u is the unique fixed point in

$$\mathcal{V}_T := \big\{v \in C\big([0,T], S^\mu_{p,q;\mathcal{B}}[E]\big)\ ;\ \|v(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]} \le M\ ,\ 0 \le t \le T\big\}$$

of the contraction $\mathcal{K} : \mathcal{V}_T \to \mathcal{V}_T$ given by

$$\mathcal{K}(v)(t) := U_B(t,0)u^0 + \int_0^t U_B(t,s)G\big(s, v(s)\big)\ \mathrm{d}s\ , \quad 0 \le t \le T\ , \quad v \in \mathcal{V}_T\ .$$

Defining then $u_0 := u^0 \in \mathcal{V}_T$ and $u_{k+1} := \mathcal{K}(u_k) \in \mathcal{V}_T$ for $k \in \mathbb{N}$, we obtain a sequence which converges in $\mathcal{V}_T$ towards u and satisfies $u_k(t) \ge 0$ for $0 \le t \le T$ and $k \in \mathbb{N}$ in view of (5.6) and Theorem 3.4 combined with [4, II. Remark 2.1.2(d)]. Since $L_p^+[E]$ is closed in $L_p[E]$, we conclude $u(t) \ge 0$ for $0 \le t \le T$. Next put

$$T^* := \sup\big\{\tau \in \dot{J}(u^0)\ ;\ u(t) \ge 0\ ,\ 0 \le t \le \tau\big\}$$

and assume $T^* < \sup J(u^0)$. Clearly $u(T^*) \ge 0$, so a repetition of the above arguments yields a contradiction. Thus $u(t) \ge 0$ for all $t \in J(u^0)$.

(ii) If $n/2p < \mu \le n/p < 2$ with $\mu \ne 1 + 1/p$, we deduce the assertion from (i), Corollary 3.5, and the continuous dependence of $u(\cdot; u^0)$ on the initial value u^0.

(iii) For the remainder of the proof, we write

$$u_p := u_p(\cdot; u^0) \in C\big(J_p(u^0), S^\mu_{p,q;\mathcal{B}}[E]\big)$$

for the maximal $L_p[E]$ solution to $(CP)_{u^0}$ with initial value $u^0 \in S^\mu_{p,q;\mathcal{B}}[E]$. In the following, we say that $P(\alpha)$ *is true* for a given $\alpha \in [2,4]$, provided

$$u_p(t; u^0) \ge 0\ , \quad t \in J_p(u^0)\ , \quad u^0 \in S^{\mu_1 +}_{p,q;\mathcal{B}}[E]\ ,$$

whenever

$$p \in (1,\infty)\ , \quad n < \alpha p\ , \quad \mu \in (n/2p\,,\, 2) \setminus \{1 + 1/p\}\ .$$

The goal is then to verify $P(4)$. First we claim that $P(\alpha)$ implies $P(2+\alpha/2)$ for $\alpha \in [2,4)$. To see this, let $\alpha \in [2,4)$ be such that $P(\alpha)$ is true and fix

$$p \in (1,\infty) \quad \text{with} \quad \alpha \le n/p < 2 + \alpha/2\ . \tag{5.7}$$

Provided $\max\big\{1 + 1/p\,,\, n/p - \alpha/2\big\} < \mu < 2$, we can choose $\varepsilon > 0$ small such that $\mu - n/p > \eta - n/\varrho$ for $\varrho := n/\alpha + \varepsilon$ and $\eta := n/2\varrho + \varepsilon$. Section 2 entails then $S^{\mu}_{p,q;\mathcal{B}}[E] \hookrightarrow S^{\eta}_{\varrho,q;\mathcal{B}}[E]$. Thus, given $u^0 \in S^{\mu_1+}_{p,q;\mathcal{B}}[E]$, Theorem 5.1 yields solutions

$$u_p = u_p(\cdot;u^0) \in C\big(J_p(u^0), S^{\mu}_{p,q;\mathcal{B}}[E]\big)$$

and

$$u_\varrho = u_\varrho(\cdot;u^0) \in C\big(J_\varrho(u^0), S^{\eta}_{\varrho,q;\mathcal{B}}[E]\big)$$

both satisfying $(CP)_{u^0}$. Furthermore, Lemma 3.2 guarantees that u_p and u_ϱ are mild solutions of $(CP)_{u^0}$ in $L_\varrho[E]$; that is, both satisfy the fixed-point equation

$$v(t) = U_{A_\varrho}(t,0)u^0 + \int_0^t U_{A_\varrho}(t,s)L\big(s,v(s)\big)\,\mathrm{d}s$$

on their domains of definition. Taking into account Theorem 2.1, Proposition 4.2, and [4, II. Lemma 5.1.3], we may apply Gronwall's inequality to deduce

$$u_p(t) = u_\varrho(t) \geq 0\ , \quad t \in J_p(u^0) \cap J_\varrho(u^0)\ ,$$

where positivity stems from the validity of $P(\alpha)$. A contradiction argument as in the last step of (i) entails then $u_p(t) \geq 0$ for $t \in J_p(u^0)$.

Now, if $\mu \in (n/2p\,,\,2) \setminus \{1+1/p\}$ is arbitrary and p still satisfies (5.7), we deduce positivity from the previous consideration by a density argument as in (ii). Therefore, $P(\alpha)$ indeed implies $P(2+\alpha/2)$ for $\alpha \in [2,4)$.

(iv) Finally, for $j \in \mathbb{N}$ put $\alpha_j := 4 - 2^{1-j} \nearrow 4$. Owing to (i) and (ii), $P(\alpha_0)$ is true. Applying (iii), we inductively obtain that also $P(\alpha_j)$ is true for $j \geq 1$. Obviously, this proves the theorem. □

5.4. **Global existence.** In this concluding subsection we give sufficient conditions for global existence. Roughly, solutions to $(CP)_{u^0}$ exist globally either if diffusion is independent of droplet size or if space dimension equals 1. We first focus on the former case.

Throughout we assume that

the hypotheses of Theorem 5.1 *as well as* (5.2), (5.3), *and* (5.5) *are satisfied.* (5.8)

Theorem 5.5. *Let* (5.8) *be valid. Then, given any $u^0 \in S^{\mu,+}_{p,q;\mathcal{B}}[E]$, the solution $u = u(\cdot;u^0)$ exists globally, that is, $J(u^0) = \mathbb{R}^+$, provided one of the following conditions holds:*

(i) *for each $T > 0$ there exists $\mathtt{C}(T) > 0$ such that*

$$\|u(t)\|_{L_\infty[L_1(Y)]} \leq \mathtt{C}(T)\ , \quad t \in J(u^0) \cap [0,T]\ ;$$

(ii) *there exists* $k \in C(\mathbb{R}^+)$ *with*

$$K(t,x;y,y') \le k(t)yy' \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad a.e.\ y,y' \in Y \ ,$$

and for each $T > 0$ *there exists* $\mathtt{C}(T) > 0$ *such that*

$$\int_Y yu(t,x,y)\ \mathrm{d}y \le \mathtt{C}(T) \ , \quad a.e.\ x \in \Omega \ , \quad t \in J(u^0) \cap [0,T] \ . \tag{5.9}$$

Proof. Let $T > 0$ be arbitrary and set $J_T := J(u^0) \cap [0,T]$. In the following we denote by $c(T)$ various constants which depend on T but not on other crucial variables. Temporarily, fix $t \in J_T$ and $x \in \Omega$ such that $u(t,x) = u(t,x,\cdot) \in E^+$ and such that

$$\int_Y u(t,x,y)\ \mathrm{d}y \le \mathtt{C}(T)$$

in case (i) or such that (5.9) holds in case (ii). For $v \in E$ we put in case (i)

$$(\mathcal{N}v)(y) := \begin{cases} v(y) \ , & y \in Y \ , \\ 0 \ , & y \in \mathbb{R} \setminus Y \ , \end{cases}$$

and in case (ii) we define

$$(\mathcal{N}v)(y) := \begin{cases} yv(y) \ , & y \in Y \ , \\ 0 \ , & y \in \mathbb{R} \setminus Y \ . \end{cases}$$

Then, it readily follows in both of the cases (i) and (ii) that

$$\begin{aligned} \big\|L_{\mathsf{b}}\big(t,u(t)\big)(x)\big\|_E + \big\|L_{\mathsf{c}}^3\big(t,u(t)\big)(x)\big\|_E + \big\|L_{\mathsf{s}}^2\big(t,u(t)\big)(x)\big\|_E \\ \le c(T)\big(1 + \|\mathcal{N}u(t,x)\|_{L_1(Y)}\big)\|u(t,x)\|_E \ , \end{aligned} \tag{5.10}$$

where $L_h^j(t,v) := L_h^j(t,v,v)$ is defined as in Section 4. Due to Young's inequality we have

$$\begin{aligned} \big\|L_{\mathsf{c}}^1\big(t,u(t)\big)(x)\big\|_E &\le c(T)\Big(\int_0^{y_0} \Big| \int_0^y K(t,x;y',y-y') \\ &\qquad \times u(t,x,y')u(t,x,y-y')\ \mathrm{d}y'\Big|^2\ \mathrm{d}y\Big)^{1/2} \\ &\le c(T)\big\|\big(\mathcal{N}u(t,x)\big) * \big(\mathcal{N}u(t,x)\big)\big\|_{L_2(\mathbb{R})} \le c(T)\|\mathcal{N}u(t,x)\|_{L_1(Y)}\|\mathcal{N}u(t,x)\|_E \ , \end{aligned} \tag{5.11}$$

where $f * g$ denotes the convolution of f and g. Moreover, applying Jensen's inequality, Fubini's theorem, and Young's inequality—in this order—we derive

$$\big\|L_{\mathsf{c}}^2\big(t,u(t)\big)(x)\big\|_E \le c(T)\Big(\int_0^{y_0} \Big| \int_y^{y_0} \beta_{\mathsf{c}}(t,x;y',y)$$

$$\times \left[\big(\mathcal{N}u(t,x)\big) * \big(\mathcal{N}u(t,x)\big)\right](y')\ \mathrm{d}y'\Big|^2\ \mathrm{d}y\Big)^{1/2} \tag{5.12}$$
$$\le c(T)\Big(\int_0^{y_0}\int_y^{y_0} |\beta_{\mathsf{c}}(t,x;y',y)|^2 \big|\left[\big(\mathcal{N}u(t,x)\big) * \big(\mathcal{N}u(t,x)\big)\right](y')\big|^2 \mathrm{d}y'\mathrm{d}y\Big)^{1/2}$$
$$\le c(T)\big\|\big(\mathcal{N}u(t,x)\big) * \big(\mathcal{N}u(t,x)\big)\big\|_{L_2(\mathbb{R})} \le c(T)\|\mathcal{N}u(t,x)\|_{L_1(Y)}\,\|\mathcal{N}u(t,x)\|_E,$$

where we additionally used that $\beta_{\mathsf{c}} \in C(\mathbb{R}^+, BUC[F_{\mathsf{b}}])$. Analogously we deduce

$$\big\|L_{\mathsf{s}}^1\big(t,u(t)\big)(x)\big\|_E \le c(T)\|\mathcal{N}u(t,x)\|_{L_1(Y)}\,\|\mathcal{N}u(t,x)\|_E\ . \tag{5.13}$$

Since $\|\mathcal{N}u(t,x)\|_{L_1(Y)} \le \mathsf{C}(T)$ by the choice of $x \in \Omega$, estimates (5.10)–(5.13) entail

$$\big\|L\big(t,u(t)\big)(x)\big\|_E \le c(T)\|u(t,x)\|_E\ ,$$

so that we end up with

$$\big\|L\big(t,u(t)\big)\big\|_{L_p[E]} \le c(T)\|u(t)\|_{L_p[E]}\ ,\quad t \in J_T\ .$$

Due to Theorem 2.1 and [4, II. Lemma 5.1.3], we have

$$\|U_{A_p}(t,s)\|_{\mathcal{L}(L_p[E],S^{\mu}_{p,q;\mathcal{B}}[E])} \le c(T)(t-s)^{-\mu/2}\ ,\quad 0 \le s < t \le T\ ,$$

whence

$$\begin{aligned}\|u(t)\|_{S^{\mu}_{p,q;\mathcal{B}}[E]} &\le \|U_{A_p}(t,0)u^0\|_{S^{\mu}_{p,q;\mathcal{B}}[E]}\\ &\qquad + \int_0^t \|U_{A_p}(t,s)\|_{\mathcal{L}(L_p[E],S^{\mu}_{p,q;\mathcal{B}}[E])}\big\|L\big(s,u(s)\big)\big\|_{L_p[E]}\ \mathrm{d}s\\ &\le c(T)\|u^0\|_{S^{\mu}_{p,q;\mathcal{B}}[E]} + c(T)\int_0^t (t-s)^{-\mu/2}\|u(s)\|_{L_p[E]}\ \mathrm{d}s\end{aligned}$$

for $t \in J_T$. The embedding $S^{\mu}_{p,q;\mathcal{B}}[E] \hookrightarrow L_p[E]$ and Gronwall's inequality imply then (5.1). □

Remark 5.6. Observe that we have proven in (5.10)–(5.13)

$$\big\|L(t,v)(x)\big\|_E \le c(T)\big(1+\|v(x)\|_{L_1(Y)}\big)\|v(x)\|_E\ ,\quad x \in \Omega\ ,\quad 0 \le t \le T\ ,$$

for $T > 0$ and $v \in E^{\Omega}$.

The next corollary specifies conditions ensuring the a priori estimates required in Theorem 5.5.

Corollary 5.7. *Suppose* (5.8), *and let the diffusion coefficient be independent of $y \in Y$ and $t \in \mathbb{R}^+$. Then $J(u^0) = \mathbb{R}^+$ for $u^0 \in S^{\mu,+}_{p,q;\mathcal{B}}[E]$ provided that one of the following conditions is valid:*

(i) *u^0 belongs to $L_\infty[L_1(Y)]$;*

(ii) u^0 *satisfies*

$$\text{ess-sup}_{x\in\Omega} \int_Y y u^0(x,y)\ \mathrm{d}y < \infty,$$

and there exists $k \in C(\mathbb{R}^+)$ *such that*

$$K(t,x;y,y') \le k(t)yy',\ (t,x) \in \mathbb{R}^+ \times \Omega,\ \textit{for almost every } y,y' \in Y. \quad (5.14)$$

Proof. Due to Theorems 5.1 and 5.4 we know that

$$u = u(\cdot;u^0) \in C^1\big(\dot{J}(u^0), L_p[E]\big) \cap C\big(\dot{J}(u^0), H^2_{p,\mathcal{B}}[E]\big)$$

is nonnegative. Using the facts that the smooth $C_c(Y)$-valued functions defined on $\bar{\Omega}$ form a dense subspace of $H^2_q[L_q(Y)] \doteq W^2_q[L_q(Y)]$ and that $L_q[L_q(Y)] = L_q\big(Y, L_q(\Omega)\big)$, we obtain from (5.2), (5.3), and [41, Lemma 2.6] that the function w, given by

$$w(s,x) := \int_Y y u(s,x,y)\ \mathrm{d}y\ , \quad \text{a.e. } x \in \Omega\ , \quad s \in J(u^0)\ ,$$

solves the problem

$$\dot{w} - d\Delta w = 0\ , \quad \partial_\nu w = 0$$

in $L_q(\Omega)$, where $q := \min\{p,2\}$. Since the scalar-valued Laplace operator subject to Neumann boundary conditions generates a semigroup of contractions on $L_\infty(\Omega)$ (see [32]), we conclude in both of the cases (i) and (ii)

$$\|w(t)\|_{L_\infty(\Omega)} \le \|w(0)\|_{L_\infty(\Omega)} < \infty\ , \quad t \in J(u^0)\ . \quad (5.15)$$

Therefore, (5.9) holds in case (ii), so we may focus on case (i) in the following. Fix $T > 0$ arbitrarily and put $J_T := J(u^0) \cap [0,T]$. Defining

$$v(t,x) := \int_Y u(t,x,y)\ \mathrm{d}y\ , \quad \text{a.e. } x \in \Omega\ , \quad t \in J_T\ ,$$

and recalling $u(t,x,\cdot) \in E^+$ for $t \in J_T$ and almost every $x \in \Omega$, we deduce that v satisfies $\partial_\nu v = 0$ and

$$\dot{v}(t) - d\Delta v(t) = \int_Y L\big(t,u(t)\big)\ \mathrm{d}y \le c(T)\big(1 + \|w(t)\|_{L_\infty(\Omega)}\big)v(t) \le \kappa v(t)$$

almost everywhere in Ω for $0 \le t \le T$ and some $\kappa := \kappa(T) > 0$, whereby the first inequality is due to (5.4) (see [41, Lemma 2.6] and the proof of [41, Theorem 2.9(ii)]) and the second is due to (5.15). This implies

$$\|u(t)\|_{L_\infty[L_1(Y)]} = \|v(t)\|_{L_\infty(\Omega)} \le e^{T\kappa}\|v(0)\|_{L_\infty(\Omega)} < \infty\ , \quad t \in J_T\ ,$$

which proves the claim. $\square$

Remark 5.8. Note that $S^{\mu}_{p,q;\mathcal{B}}[E] \hookrightarrow L_\infty[L_1(Y)]$ if $\mu > n/p$, so (i) of Corollary 5.7 holds true in this case.

In order to discuss global existence for diffusion coefficients depending on droplet size, let us first recall some facts on interpolation-extrapolation theory. Concerning more detailed information on this subject, we refer to [4].

For the remainder we assume

$$d \in C\big(\bar{Y}, (0,\infty)\big) \quad \text{and} \quad 1 < p, q < \infty \ . \tag{5.16}$$

Put $\mathbb{E}_0 := L_p[E]$ and $\mathbb{E}_1 := H^2_{p,\mathcal{B}}[E]$. Then $\mathbb{E}_1$ is dense in the reflexive space $\mathbb{E}_0$ and

$$\mathbb{A}_0 := A_p = A_p[d] \in \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) \ .$$

Choose $\omega > 0$ such that the spectrum of $\omega + \mathbb{A}_0$ is contained in $[\mathrm{Re} z > 0]$, and set

$$\mathbb{E}_{-1} := \big(\mathbb{E}_0, \|(\omega + \mathbb{A}_0)^{-1} \cdot \|_{\mathbb{E}_0}\big)^{\sim},$$

where $\big(\,\cdot\,\big)^{\sim}$ means completion. Denoting by $\mathbb{A}_{-1}$ the closure of $\mathbb{A}_0$ in $\mathbb{E}_{-1}$, we have $\mathbb{A}_{-1} \in \mathcal{H}(\mathbb{E}_0, \mathbb{E}_{-1})$. We then set for $\theta \in (0,1)$

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,q} & \text{if} \quad \big\{S^{\mu}_{p,q}[E]\,;\, \mu \neq 0\big\} = \big\{B^{\mu}_{p,q}[E]\,;\, \mu \neq 0\big\}\ , \\ [\cdot,\cdot]_\theta & \text{if} \quad \big\{S^{\mu}_{p,q}[E]\,;\, \mu \neq 0\big\} = \big\{H^{\mu}_{p}[E]\,;\, \mu \neq 0\big\}\ . \end{cases}$$

Given $k \in \{0,-1\}$ and $\theta \in (0,1)$, define $\mathbb{E}_{k+\theta} := (\mathbb{E}_k, \mathbb{E}_{k+1})_\theta$ and denote by $\mathbb{A}_{k+\theta}$ the $\mathbb{E}_{k+\theta}$-realization of $\mathbb{A}_k$. Furthermore, put for $\theta \in (0,1)$

$$(\cdot,\cdot)^\sharp_\theta := \begin{cases} (\cdot,\cdot)_{\theta,q'} & \text{if} \quad (\cdot,\cdot)_\theta = (\cdot,\cdot)_{\theta,q} \\ [\cdot,\cdot]_\theta & \text{if} \quad (\cdot,\cdot)_\theta = [\cdot,\cdot]_\theta \end{cases}$$

and $\mathbb{E}^\sharp_0 := (\mathbb{E}_0)' \doteq L_{p'}[E]$. Then Lemma 3.6 guarantees

$$\mathbb{A}^\sharp_0 := (\mathbb{A}_0)' = (A_p)' = A_{p'},$$

and hence $\mathbb{E}^\sharp_1 := D\big(\mathbb{A}^\sharp_0\big) \doteq H^2_{p',\mathcal{B}}[E]$. Choose $\omega^\sharp > 0$ similarly as before and put

$$\mathbb{E}^\sharp_{-1} := \big(\mathbb{E}^\sharp_0, \big\|\big(\omega^\sharp + \mathbb{A}^\sharp_0\big)^{-1} \cdot \big\|_{\mathbb{E}^\sharp_0}\big)^{\sim}$$

and again $\mathbb{E}^\sharp_{k+\theta} := \big(\mathbb{E}^\sharp_k, \mathbb{E}^\sharp_{k+1}\big)^\sharp_\theta$ for $k \in \{0,-1\}$ and $\theta \in (0,1)$.

As the next proposition shows, each $-\mathbb{A}_\alpha$ is the generator of an analytic semigroup in $\mathbb{E}_\alpha$ and the latter can be further characterized.

Proposition 5.9. *(i) It holds that* $\mathbb{A}_\alpha \in \mathcal{H}(\mathbb{E}_{\alpha+1}, \mathbb{E}_\alpha)$ *for* $-1 \le \alpha \le 0$ *and* $\mathbb{A}_\alpha \supset \mathbb{A}_\beta$ *for* $-1 \le \alpha < \beta \le 0$.

(*ii*) *It holds that*

$$\mathbb{E}_\theta \doteq \begin{cases} S^{2\theta}_{p,q;\mathcal{B}}[E] & \textit{if} \quad 1+1/p \ < \ 2\theta \ \le \ 2 \ , \\ S^{2\theta}_{p,q}[E] & \textit{if} \quad -1+1/p \ < \ 2\theta \ < \ 1+1/p \ , \\ \big[S^{-2\theta}_{p',q'}[E]\big]' & \textit{if} \quad -2+1/p \ < \ 2\theta \ \le \ -1+1/p \ , \\ \big[S^{-2\theta}_{p',q';\mathcal{B}}[E]\big]' & \textit{if} \quad -2 \ \le \ 2\theta \ < \ -2+1/p \ . \end{cases}$$

Proof. Since (i) follows from [4], it remains to prove (ii). From Theorem 2.1 we obtain

$$\mathbb{E}_\theta \doteq S^{2\theta}_{p,q;\mathcal{B}}[E] \ , \quad 2\theta \in [0,2] \setminus \{1+1/p\} \ ,$$

and

$$\mathbb{E}^\sharp_\theta \doteq S^{2\theta}_{p',q';\mathcal{B}}[E] \ , \quad 2\theta \in [0,2] \setminus \{1+1/p'\} \ .$$

According to [4, V. Theorem 1.5.12], the dual space $(\mathbb{E}_\theta)'$ of $\mathbb{E}_\theta$ (with respect to the duality pairing induced by the L_p-duality pairing) coincides with $\mathbb{E}^\sharp_{-\theta}$ except for equivalent norms. Hence $\mathbb{E}_\theta \doteq \big(\mathbb{E}^\sharp_{-\theta}\big)'$ due to the fact that $\mathbb{E}_\theta$ is reflexive (observe that $(\cdot,\cdot)_\theta$ is admissible), and finally

$$\mathbb{E}_\theta \doteq \big[S^{-2\theta}_{p',q';\mathcal{B}}[E]\big]' \ , \quad 2\theta \in [-2,0] \setminus \{-2+1/p\} \ .$$

Recalling (2.10) and the definition of the spaces $S^{2\theta}_{p,q;\mathcal{B}}[E]$ for $2\theta > 0$, the assertion is evident. $\square$

Corollary 5.10. $L_1[E] \hookrightarrow \mathbb{E}_\theta$ *provided* $n(1-1/p) < -2\theta < 2-1/p$.

Proof. From $-2\theta > n/p'$ it follows that $S^{-2\theta}_{p',q'}[E] \hookrightarrow C(\bar{\Omega},E)$, and this embedding is dense. Thus, for each $w \in L_1[E]$,

$$\Big| \int_\Omega \big(w(x)|v(x)\big)_E \,\mathrm{d}x\Big| \le c\|w\|_{L_1[E]}\,\|v\|_{S^{-2\theta}_{p',q'}[E]} \ , \quad v \in S^{-2\theta}_{p',q'}[E] \ ,$$

whence $w \in \big[S^{-2\theta}_{p',q'}[E]\big]'$. Part (ii) of Proposition 5.9 entails then the assertion. $\square$

Now we can state the theorem on global existence for droplet-size-dependent diffusion coefficients in case of $n = 1$.

Theorem 5.11. *Suppose* $n = 1$, *and let* (5.8) *and* (5.16) *be satisfied. Moreover, suppose*

(i) *for* $(t,x) \in \mathbb{R}^+ \times \Omega$ *and almost every* $y + y' \in Y$ *it holds that*

$$Q(t,x;y,y')\left(\int_0^{y+y'} \beta_{\mathsf{c}}(t,x;y+y',y'')\,\mathrm{d}y'' - 2\right) \le P(t,x;y,y')$$

and

(ii) *for* $(t,x)\in\mathbb{R}^+\times\Omega$ *and almost every* $y+y'\in(y_0,2y_0]$ *it holds that*

$$\int_0^{y_0}\beta_{\mathsf{s}}(t,x;y+y',y'')\ \mathrm{d}y''=2\ .$$

Then $J(u^0)=\mathbb{R}^+$ *provided* $u^0\in S^{\mu,+}_{p,q;\mathcal{B}}[E]$ *with* $\mu>1/p$.

Proof. Integrating the equation

$$\dot u-d(y)\Delta u=L\big(t,u(t)\big)\ ,\quad t\in\dot J(u^0)\ ,$$

over $\Omega\times Y$ and taking into account the Neumann boundary conditions, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\int_Y u(t)\ \mathrm{d}y\,\mathrm{d}x=\int_\Omega\int_Y L\big(t,u(t)\big)\ \mathrm{d}y\,\mathrm{d}x\ ,\quad t\in\dot J(u^0)\ .$$

In view of [41, Lemma 2.6], the positivity of $u(t)$ and hypotheses (i) and (ii) yield

$$\int_\Omega\int_Y L\big(t,u(t)\big)\ \mathrm{d}y\,\mathrm{d}x\le c(T)\int_\Omega\int_Y u(t)\ \mathrm{d}y\,\mathrm{d}x\ ,\quad t\in J_T\ ,$$

where $J_T:=J(u^0)\cap[0,T]$ for $T>0$ arbitrary. Therefore,

$$\|u(t)\|_{L_1[L_1(Y)]}\le c(T)\ ,\quad t\in J_T\ . \tag{5.17}$$

We may assume $1/p<\mu<1+1/p$. Fixing $2\zeta\in(-2+\mu,-1+1/p)$, Proposition 5.9 yields $\mathbb{A}_\zeta\in\mathcal{H}(\mathbb{E}_{\zeta+1},\mathbb{E}_\zeta)$. Let $0<\vartheta<\eta<1$ be with

$$-2+1/p<2\zeta+2\vartheta<-1+1/p<\mu=2\zeta+2\eta<2\zeta+2,$$

and choose $\varepsilon>0$ small. From Corollary 5.10 and [4, V. Theorem 1.5.3] we deduce

$$L_1[E]\hookrightarrow\mathbb{E}_{\zeta+\vartheta}\hookrightarrow(\mathbb{E}_\zeta,\mathbb{E}_{\zeta+1})_{\vartheta-\varepsilon}=:\mathtt{E}_{\vartheta-\varepsilon} \tag{5.18}$$

and

$$\mathtt{E}_{\eta+\varepsilon}:=(\mathbb{E}_\zeta,\mathbb{E}_{\zeta+1})_{\eta+\varepsilon}\hookrightarrow\mathbb{E}_{\mu/2}\doteq S^\mu_{p,q;\mathcal{B}}[E]\ . \tag{5.19}$$

Furthermore, Remark 5.6 entails that, for $t\in J_T$,

$$\begin{aligned}\big\|L\big(t,u(t)\big)\big\|_{L_1[E]}&\le c(T)\big(1+\|u(t)\|_{L_1[L_1(Y)]}\big)\|u(t)\|_{L_\infty[E]}\\&\le c(T)\|u(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]}\ ,\end{aligned}$$

where the second inequality stems from (5.17) and $\mu>1/p$. Due to (5.18) we thus conclude

$$\big\|L\big(t,u(t)\big)\big\|_{\mathtt{E}_{\vartheta-\varepsilon}}\le c(T)\|u(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]}\ ,\quad t\in J_T\ . \tag{5.20}$$

Since $\mathbb{E}_0 \hookrightarrow \mathbb{E}_\zeta$ and $\mathbb{E}_1 \hookrightarrow \mathbb{E}_{\zeta+1}$, we see that

$$u \in C(J_T, \mathbb{E}_\zeta) \cap C(\dot{J}_T, \mathbb{E}_{\zeta+1}) \cap C^1(\dot{J}_T, \mathbb{E}_\zeta)$$

solves the Cauchy problem

$$\dot{u} + \mathbb{A}_\zeta u = L(t,u) \quad \text{in} \quad \mathbb{E}_\zeta \ .$$

Thus, owing to (5.19), (5.20), and [4, V. Theorem 1.5.3 and II. Lemma 5.1.3], it follows that

$$\begin{aligned} \|u(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]} &\leq \|e^{-t\mathbb{A}_\zeta} u^0\|_{S^\mu_{p,q;\mathcal{B}}[E]} \\ &\qquad + c\int_0^t \|e^{-(t-s)\mathbb{A}_\zeta}\|_{\mathcal{L}(\mathtt{E}_{\vartheta-\varepsilon},\mathtt{E}_{\eta+\varepsilon})} \big\|L\big(s,u(s)\big)\big\|_{\mathtt{E}_{\vartheta-\varepsilon}} \, \mathrm{d}s \\ &\leq c(T,u^0)t^{-\varepsilon} + c(T)\int_0^t (t-s)^{-\lambda} \|u(s)\|_{S^\mu_{p,q;\mathcal{B}}[E]} \, \mathrm{d}s \end{aligned}$$

for $t \in \dot{J}_T$, where $\lambda := 3\varepsilon + \eta - \vartheta \in (0,1)$. The singular Gronwall's inequality (see [4, II. Corollary 3.3.2]) entails then for $t_0 \in \dot{J}(u^0)$

$$\|u(t)\|_{S^\mu_{p,q;\mathcal{B}}[E]} \leq c(T,t_0) \ , \quad t \in J(u^0) \cap [t_0, T] \ .$$

Hence, Theorem 5.1 leads to the assertion. □

Remark 5.12. Observe that hypothesis (ii) of Theorem 5.11 indicates binary scattering; i.e., each scattering event results in exactly two daughter droplets. Also note that hypothesis (i) of Theorem 5.11 holds provided that collisional breakage is a binary mechanism, or if it is absent, that is, if $Q \equiv 0$.

Example 5.13. Let us consider a possible choice of kernels to illustrate our preceding results, namely if fragmentation is subject to a power-law break up. For simplicity we omit time and space dependence.

Suppose $P, Q \in L^+_\infty(Y \times Y)$ are symmetric, satisfy $0 \leq P + Q \leq 1$, and that there exists $\mathtt{y} \in Y$ such that $Q(y,y') = 0$ for almost every $0 < y + y' < \mathtt{y}$. Define then

$$\begin{aligned} \gamma(y,y') &:= hy^{\alpha-\xi-1}(y')^\xi \ , && 0 < y' < y \leq y_0 \ , \\ \beta_{\mathsf{c}}(y,y') &:= (\zeta+2)y^{-1-\zeta}(y')^\zeta \ , && \mathtt{y} \leq y \leq y_0 \ , \quad 0 < y' < y \ , \\ \beta_{\mathsf{s}}(y,y') &:= (\nu+2)y_0^{-2-\nu} y (y')^\nu \ , && 0 < y' \leq y_0 < y \leq 2y_0 \ , \end{aligned}$$

with $\alpha \geq 1/2$, $0 \geq \xi, \zeta, \nu > -1/2$, and $h > 0$. Extending γ and β_{c} by zero it is easily seen that

$$(\gamma, \beta_{\mathsf{c}}, \beta_{\mathsf{s}}) \in F^+_{\mathsf{b}} \times F^+_{\mathsf{b}} \times F^+_{\mathsf{s}}$$

and that (5.2) as well as (5.3) are satisfied. Moreover, (i) of Theorem 5.11 holds provided that

$$\frac{-\zeta}{\zeta+1}\,Q(y,y') \le P(y,y')\ , \quad \text{a.e. } \mathtt{y} \le y+y' \le y_0\ ,$$

e.g. if $\zeta = 0$ (corresponding to binary shattering), and (ii) of Theorem 5.11 holds for $\nu = 0$ (binary scattering). Finally, collision rates of the form

$$K(y,y') := A + B(y+y')^\sigma + C(yy')^\tau\ , \quad y,y' \in Y\ ,$$

with $\sigma,\tau \ge 0$ and $A,B,C \ge 0$ belong to F_{c}^+ and, in addition, (5.14) is valid for $A = B = 0$ and $\tau \ge 1$.

For further examples we refer to [42].

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