SOME REMARKS ON THE ASYMPTOTIC BEHAVIOR OF THE SEMIGROUP ASSOCIATED WITH AGE-STRUCTURED DIFFUSIVE POPULATIONS

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Abstract. We consider linear age-structured population equations with diffusion. Supposing maximal regularity of the diffusion operator, we characterize the generator and its spectral properties of the associated strongly continuous semigroup. In particular, we provide conditions for stability of the zero solution and for asynchronous exponential growth.

1. Introduction

There are numerous applications of population models incorporating age and spatial structure, including – among many more – tumor growth, bacteria swarming, and epidemic models (for example, see the references in [18, 22]). In many instances, the age of individuals has an important impact on the processes determining the dynamics of the population, e.g. on birth and death processes or on spatial dispersion. To consider a simple prototype model describing the evolution of an age and spatially structured population, let $u = u(t, a, x) \geq 0$ denote the distribution density with respect to age $a \in J := [0, a_m)$ and spatial position $x \in \Omega$ of the population at time $t \geq 0$, where $a_m \in (0, \infty]$ is the maximal age and Ω is a bounded and smooth domain in $\mathbb{R}^n$. Suppose that the individuals' movement can be described by a diffusion term $\mathrm{div}_x(d(a, x)\nabla_x u)$, where the dispersal speed $d(a, x) > 0$ reflects a movement behavior depending on age class and takes into account spatial heterogeneity of the environment. Further suppose that also the birth and death rates, $b = b(a, x) \geq 0$ and $\mu = \mu(a, x) \geq 0$, respectively, depend on age $a \in J$ and spatial position $x \in \Omega$. If the individuals may not leave the spatial domain, then the dynamics of the population with initial distribution $\phi = \phi(a, x) \geq 0$ is governed by the equations

$$\partial_t u + \partial_a u = \mathrm{div}_x\big(d(a, x)\nabla_x u\big) - \mu(a, x)u\ , \qquad t > 0\,, \qquad a \in (0, a_m)\,, \qquad x \in \Omega\ , \tag{1.1}$$

$$u(t, 0, x) = \int_0^{a_m} b(a, x)u(a)\,\mathrm{d}a\ , \qquad t > 0\,, \qquad x \in \Omega\ , \tag{1.2}$$

$$\partial_\nu u(t, a, x) = 0\ , \qquad t > 0\,, \qquad a \in (0, a_m)\,, \qquad x \in \partial\Omega\ , \tag{1.3}$$

$$u(0, a, x) = \phi(a, x)\ , \qquad a \in (0, a_m)\,, \qquad x \in \Omega\ , \tag{1.4}$$

with ν denoting the outward unit normal on $\partial\Omega$. Equations like these and variants thereof, e.g. for constant or time-dependent (general second order elliptic) diffusion operators, have been investigated by many authors, for example see [5, 7, 11, 12, 14, 15, 22] and the references therein though this list is far from being complete. Our objective here is to provide an abstract functional analytic framework – based on the property of maximal regularity of the diffusion operator – which yields a suitable theory on existence and large time dynamics of solutions and that allows one to tackle nonlinear problems later on. To set the stage, let $E_0 := L_p(\Omega)$ with $p \in (1, \infty)$ be ordered by its positive cone $E_0^+ = L_p^+(\Omega)$ of functions that are nonnegative almost everywhere.

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Let $E_1 := W^2_{p,\mathcal{B}}(\Omega)$ denote the space consisting of all functions $w : \Omega \to \mathbb{R}$ in the usual Sobolev space $W^2_p(\Omega)$ satisfying the boundary condition $\partial_\nu w = 0$ on $\partial\Omega$. Setting

$$A(a)w := -\mathrm{div}_x\big(d(a,\cdot)\nabla_x w\big) + \mu(a,\cdot)w\ , \quad w \in E_1\ ,$$

we can reformulate (1.1)-(1.4) as an abstract problem of the form

$$\partial_t u + \partial_a u + A(a)\,u = 0\ , \qquad t > 0\ , \quad a \in (0,a_m)\ , \tag{1.5}$$

$$u(t,0) = \int_0^{a_m} b(a)\,u(t,a)\,\mathrm{d}a\ , \qquad t > 0\ , \tag{1.6}$$

$$u(0,a) = \phi(a)\ , \qquad a \in (0,a_m)\ , \tag{1.7}$$

where $u : \mathbb{R}^+ \times J \to E_0^+$ with $u(t,a) \in E_1$ for $(t,a) \in (0,\infty)\times(0,a_m)$. All equations are understood in the Banach lattice E_0. In the following, we shall focus our attention on this abstract form.

Recall, e.g. from [22], that a strongly continuous semigroup in $L_1(J,E_0)$ can be associated with (1.5)-(1.7) if $-A$ is independent of age and generates itself a strongly continuous semigroup on E_0, see also [5, 15]. This is derived upon formally integrating (1.5) along characteristics what gives the semigroup rather explicitly. The approach has been extended to investigate the well-posedness of models featuring nonlinearities in the operator $A = A(t,u)$ or in the birth rates $b = b(t,u)$ [16, 18].

A slightly different approach has been chosen in [14]. On employing methods for positive perturbations of semigroups it has been shown that a strongly continuous semigroup for (1.5)-(1.7) in $L_1(J,E_0)$ is obtained as the derivative of an integrated semigroup. Moreover, this strongly continuous semigroup is shown to enjoy certain compactness properties and to exhibit asynchronous exponential growth, i.e. it stabilizes as $t \to \infty$ to a one-dimensional image of the state space of initial values, after multiplication by an exponential factor in time. This result has been recovered as a particular case in [12] (also see [11]), where time-dependent birth rates have been included by means of perturbation techniques of Miyadera type. It is noteworthy that the general results of [14] apply as well to other situations than A describing spatial diffusion.

The strongly continuous semigroup from [22, 15] associated with (1.5)-(1.7) has the advantage that certain properties — like regularizing effects in the case that $-A$ generates an analytic semigroup, being of utmost importance in nonlinear equations, see [16, 18] — can be read off its formula rather easily, see Theorem 2.2 below and the subsequent remarks. The domain of the generator of this semigroup is in general not fully identified, cf. [15, 16]. The objective of the present paper is to characterize the (domain of the) infinitesimal generator of the strongly continuous semigroup associated with (1.5)-(1.7) and to investigate its spectral properties in the case that the operator $-A$ has the property of maximal L_p-regularity. This assumption is satisfied in many applications, e.g. when $-A$ is a second order elliptic differential operator in divergence form as in (1.1). Maximal regularity provides an adequate functional analytic setting for the characterization of the generator of the semigroup associated with (1.5)-(1.7) in the phase space $L_p(J,E_0)$ and its resolvent, see Theorem 2.8 below. Knowing the generator precisely, we then investigate its growth bound and derive a stability result for the trivial solution, see Theorem 3.5 below. Moreover, we provide in Theorem 3.7 a condition that implies asynchronous exponential growth of the semigroup. Finally, we give an application to time-dependent diffusion operators $A = A(t,a)$ in Proposition 2.10.

Besides a precise description of the semigroup with its generator and stability of the trivial solution, we obtain a similar result on asynchronous exponential growth as previously proved in [14, 12] by another approach which is inspired by the results in [20] that were dedicated to the non-diffusive scalar case. We shall point out, however, that the results and the approach presented herein may serve as a basis for a future investigation of qualitative aspects of solutions to models featuring nonlinearities in the diffusion part and in the age-boundary condition, i.e. for diffusion operators of the form $A = A(u)$ and birth rates $b = b(u)$, by means of linearization and perturbation techniques. Finally, from a technical point of view (and different than in many other research papers) it also seems to be worthwhile to point out that the cases of a finite or infinite maximal age a_m are treated simultaneously herein.

2. The Semigroup and its Generator

2.1. **Notation and Assumptions.** Given a closed linear operator $\mathcal{A}$ on a Banach space, we let $\sigma(\mathcal{A})$ and $\sigma_p(\mathcal{A})$ denote its spectrum and point spectrum, respectively. The essential spectrum $\sigma_e(\mathcal{A})$ of $\mathcal{A}$ consists of those spectral points λ of $\mathcal{A}$ such that the image $\mathrm{im}(\lambda-\mathcal{A})$ is not closed, or λ is a limit point of $\sigma(\mathcal{A})$, or the dimension of the kernel $\ker(\lambda-\mathcal{A})$ is infinite. The peripheral spectrum $\sigma_0(\mathcal{A})$ is defined as $\sigma_0(\mathcal{A}) := \{\lambda \in \sigma(\mathcal{A})\,;\, \mathrm{Re}\,\lambda = s(\mathcal{A})\}$, where $s(\mathcal{A}) := \sup\{\mathrm{Re}\,\lambda\,;\, \lambda\in\sigma(\mathcal{A})\}$ denotes the spectral bound of $\mathcal{A}$. The resolvent set $\mathbb{C}\setminus\sigma(\mathcal{A})$ is denoted by $\varrho(\mathcal{A})$.

Throughout E_0 is a real Banach lattice ordered by a closed convex cone E_0^+. However, we do not distinguish E_0 from its complexification in our notation as no confusion seems likely. Recall that $u\in E_0^+$ is quasi-interior if $\langle f,u\rangle>0$ for all f in the dual space E_0' with $f>0$.

Let E_1 be a densely and compactly embedded subspace of E_0. We fix $p\in(1,\infty)$, put $\varsigma := \varsigma(p) := 1-1/p$ and set

$$E_\varsigma := (E_0,E_1)_{\varsigma,p}\ , \quad E_\theta := (E_0,E_1)_\theta$$

for $\theta\in[0,1]\setminus\{1-1/p\}$ with $(\cdot,\cdot)_{\varsigma,p}$ being the real interpolation functor and $(\cdot,\cdot)_\theta$ being any admissible interpolation functor. We equip these interpolation spaces with the order naturally induced by E_0^+. Observe that E_θ embeds compactly in E_ϑ provided $0\le\vartheta<\theta\le 1$. Recall that $J=[0,a_m)$, where $a_m\in(0,\infty]$ so that J may be bounded or unbounded. We put

$$\mathbb{E}_0 := L_p(J,E_0)\ , \qquad \mathbb{E}_1 := L_p(J,E_1)\cap W_p^1(J,E_0)$$

and recall that

$$\mathbb{E}_1 \hookrightarrow BUC(J,E_\varsigma) \tag{2.1}$$

according to, e.g. [1, III.Thm.4.10.2], where BUC stands for the bounded and uniformly continuous functions. In particular, the trace $\gamma_0 u := u(0)$ is well-defined for $u\in\mathbb{E}_1$ and $\gamma_0\in\mathcal{L}(\mathbb{E}_1,E_\varsigma)$. Let $\mathbb{E}_0^+$ denote the functions in $\mathbb{E}_0$ taking almost everywhere values in E_0^+. Note that $\mathbb{E}_0$ is a Banach lattice. We further assume that

$$A\in L_\infty(J,\mathcal{L}(E_1,E_0))\ , \quad \sigma + A\in C^\rho(J,\mathcal{H}(E_1,E_0;\kappa,\nu)) \tag{2.2}$$

for some $\rho,\nu>0$, $\kappa\ge 1$, $\sigma\in\mathbb{R}$. Here $\mathcal{H}(E_1,E_0;\kappa,\nu)$ consists of all negative generators $-\mathcal{A}$ of analytic semigroups on E_0 with domain E_1 such that $\nu+\mathcal{A}$ is an isomorphism from E_1 to E_0 and

$$\kappa^{-1}\le \frac{\|(\lambda+\mathcal{A})x\|_{E_0}}{|\lambda|\,\|x\|_{E_0}+\|x\|_{E_1}}\le\kappa\ , \quad x\in E_1\setminus\{0\}\ , \quad \mathrm{Re}\,\lambda\ge\nu\ .$$

Note that A generates a parabolic evolution operator $\Pi(a,\sigma) := \Pi_A(a,\sigma)$, $0\le\sigma\le a<a_m$, on E_0 with regularity subspace E_1 according to [1, II.Cor.4.4.2] and there are $M\ge 1$ and $\varpi\in\mathbb{R}$ such that

$$\|\Pi(a,\sigma)\|_{\mathcal{L}(E_\alpha)} + (a-\sigma)^{\alpha-\beta_1}\|\Pi(a,\sigma)\|_{\mathcal{L}(E_\beta,E_\alpha)}\le Me^{-\varpi(a-\sigma)}\ , \quad 0\le\sigma\le a<a_m\ , \tag{2.3}$$

for $0\le\beta_1\le\beta<\alpha\le 1$ with $\beta_1<\beta$ if $\beta>0$, see [1, II.Lem.5.1.3]. We further assume that $\Pi(a,\sigma)$ is positive for $0\le\sigma\le a<a_m$ and that

$$\varpi>0 \quad \text{if} \quad a_m=\infty\ . \tag{2.4}$$

Moreover, we assume that

$$\begin{aligned}&\text{for each Re } \lambda>-\varpi, \text{ the operator } A_\lambda := \lambda + A \text{ has maximal } L_p\text{-regularity,}\\ &\text{that is, } (\partial_a + A_\lambda,\gamma_0):\mathbb{E}_1\to\mathbb{E}_0\times E_\varsigma \text{ is an isomorphism .}\end{aligned} \tag{2.5}$$

Let the birth rate b be such that

$$b\in L_\infty(J,\mathcal{L}(E_\theta))\cap L_{p'}(J,\mathcal{L}(E_\theta))\ , \qquad b(a)\in\mathcal{L}_+(E_0)\ , \quad a\in J\ , \tag{2.6}$$

for $\theta\in[0,1]$, where p' is the dual exponent of p. We also assume that

$$b(a)\Pi(a,0)\in\mathcal{L}_+(E_0) \text{ is irreducible for } a \text{ in a subset of } J \text{ of positive measure}\ . \tag{2.7}$$

Some of the assumptions above are redundant if $a_m<\infty$. For instance, if $a_m<\infty$ and

$$A\in C^\rho([0,a_m],\mathcal{L}(E_1,E_0))$$

is such that $-A(a)$ generates an analytic semigroup on E_0 for each $a\in J$, then (2.2) holds. We shall furthermore point out that not all assumptions will be needed in this strength but are imposed for the sake of simplicity. In particular, if μ being a real-valued, nonnegative continuous function and $A(a)$ is as in the introduction, then it suffices that the diffusion part $A(a)-\mu(a)$ satisfies (2.2) for what follows by keeping in mind that

$$\Pi(a,\sigma)=e^{-\int_\sigma^a\mu(r)\mathrm{d}r}U(a,\sigma)$$

with U denoting the evolution operator associated with $A-\mu$. Also, (2.6) is not required for the whole range of $\theta\in[0,1]$.

We remark that the assumptions above are satisfied in many applications with A describing spatial diffusion, for example see [17, Sect.3], [19, Sect.3]. For details about parabolic evolution operators and operators having maximal regularity we refer the reader, e.g., to [1]. A summary on positive operators in ordered Banach spaces can be found e.g. in [3].

Due to (2.5), the operator $A_\lambda=\lambda+A$ has maximal L_p-regularity on J for each $\lambda\in\mathbb{C}$ provided $\mathrm{Re}\,\lambda>-\varpi$ if $a_m=\infty$ and it generates a parabolic evolution operator

$$\Pi_\lambda(a,\sigma):=e^{-\lambda(a-\sigma)}\,\Pi(a,\sigma)\ ,\quad 0\le s\le a<a_m\ ,$$

on E_0. Consequently, the unique solution $\phi\in\mathbb{E}_1$ to

$$\partial_a\phi+A_\lambda(a)\phi=f(a)\ ,\quad a\in(0,a_m)\ ,\qquad \phi(0)=\phi_0$$

for $\phi_0\in E_\varsigma$ and $f\in\mathbb{E}_0$ is given by

$$\phi(a)=\Pi_\lambda(a,0)\phi_0+\int_0^a\Pi_\lambda(a,s)\,f(s)\,\mathrm{d}s\ ,\quad a\in J\ .$$

In particular, $\Pi_\lambda(\cdot,0)\in\mathcal{L}(E_\varsigma,\mathbb{E}_1)$ for $\lambda\in\mathbb{C}$ with $\mathrm{Re}\,\lambda>-\varpi$ if $a_m=\infty$.

2.2. **The Semigroup and its Generator.** On integrating (1.5) along characteristics we formally derive that the solution $[S(t)\phi](a):=u(t,a)$ to (1.5)-(1.7) is given by

$$\big[S(t)\phi\big](a)\ :=\ \begin{cases}\Pi(a,a-t)\,\phi(a-t)\ , & 0\le t\le a<a_m\ ,\\ \Pi(a,0)\,B_\phi(t-a)\ , & 0\le a<a_m\,,\ t>a\ ,\end{cases} \tag{2.8}$$

where $B_\phi:=u(\cdot,0)$ satisfies according to (1.6) the Volterra equation

$$B_\phi(t)\ =\ \int_0^t h(a)\,b(a)\,\Pi(a,0)\,B_\phi(t-a)\,\mathrm{d}a+\int_0^{a_m-t}h(a)\,b(a+t)\,\Pi(a+t,a)\,\phi(a)\,\mathrm{d}a\ ,\quad t\ge0\ , \tag{2.9}$$

with cut-off function $h(a):=1$ if $a\in(0,a_m)$ and $h(a):=0$ otherwise. Note that

$$B_\phi(t)\ =\ \int_0^{a_m}b(a)\,\big[S(t)\phi\big](a)\,\mathrm{d}a\ ,\quad t\ge0\ . \tag{2.10}$$

To make the formal integration rigorous, we first observe:

Lemma 2.1. *There exists a mapping* $[\phi\mapsto B_\phi]\in\mathcal{L}\big(\mathbb{E}_0,C(\mathbb{R}^+,E_0)\big)$ *such that* B_ϕ *is the unique solution to* (2.9)*. If* $\phi\in\mathbb{E}_0^+$*, then* $B_\phi(t)\in E_0^+$ *for* $t\ge0$*. Given* $\theta\in[0,1]$*, there is* $N:=N(\theta)>0$ *such that*

$$\|B_\phi(t)\|_{E_\theta}\ \le\ N\,t^{-\theta}e^{(-\varpi+\zeta(\theta))t}\,\|\phi\|_{\mathbb{E}_0}\ ,\quad t>0\ , \tag{2.11}$$

where $\zeta(\theta):=(1+\theta)M\|b\|_{L_\infty(J,\mathcal{L}(E_\theta))}$*.*

Proof. The proof is straightforward by standard arguments, similar statements are found in [22, Thm.4] and [15, Lem.2.1]. We only note that one obtains, for $t>0$, on applying (2.3) to (2.9) and on using (2.6),

$$e^{\varpi t}\|B_\phi(t)\|_{E_\theta} \le M\|b\|_{L_\infty(J,\mathcal{L}(E_\theta))}\int_0^t e^{\varpi a}\,\|B_\phi(a)\|_{E_\theta}\,\mathrm{d}a + M\|b\|_{L_{p'}(J,\mathcal{L}(E_\theta))}\|\phi\|_{\mathbb{E}_0}t^{-\theta}$$

and thus (2.11) follows from the singular Gronwall's inequality [1, II.Cor.3.3.2]. □

Along the lines of [22, Thm.4] (for the case $p=1$) and on using (2.3) and (2.11) (also see [15]) one easily proves the following:

Theorem 2.2. *$\{S(t)\,;\,t\ge 0\}$ given in (2.8) is a strongly continuous positive semigroup in $\mathbb{E}_0$ with*

$$\sup_{t\ge 0} e^{t(\varpi-\zeta)}\|S(t)\|_{\mathcal{L}(\mathbb{E}_0)}<\infty\ ,$$

where $\zeta:=\zeta(0)=M\|b\|_{L_\infty(J,\mathcal{L}(E_0))}$.

To include the case of time-dependent diffusion operators let us also observe that the same calculations based on (2.11) yield:

Remark 2.3. *If in (2.3) we have*

$$\|\Pi(a,\sigma)\|_{\mathcal{L}(E_0)}\le e^{\eta(a-\sigma)}\ ,\quad 0\le\sigma\le a<a_m\ ,$$

for some $\eta\in\mathbb{R}$, then there is $\xi:=\xi(\eta)\in\mathbb{R}$ such that $\|S(t)\|_{\mathcal{L}(\mathbb{E}_0)}\le e^{\xi t}$ for $t\ge 0$.

Though we shall not use it in the following let us also note that the semigroup $\{S(t)\,;\,t\ge0\}$ inherits regularizing properties from the parabolic evolution operator stated in (2.3) , e.g. there holds

$$\|S(t)\phi\|_{L_p(J,E_\theta)}\le c(\theta)\,t^{-\theta}\,e^{t(-\varpi+\zeta(\theta))}\,\|\phi\|_{\mathbb{E}_0}\ ,\quad t>0\ ,\quad \phi\in\mathbb{E}_0\ ,\quad \theta\in[0,1/p)\ .$$

Let $-\mathbb{A}$ denote the generator of the semigroup $\{S(t)\,;\,t\ge0\}$. Based on the assumption of maximal regularity of the operator A in (1.5), we now fully characterize $-\mathbb{A}$. First, recall that $\{\lambda\in\mathbb{C}\,;\,\mathrm{Re}\,\lambda>\omega(-\mathbb{A})\}$ is a subset of the resolvent set $\varrho(-\mathbb{A})$, where the growth bound $\omega(-\mathbb{A})$ is given by

$$\omega(-\mathbb{A}):=\lim_{t\to\infty}\frac{1}{t}\log\|S(t)\|\ .$$

Note that Theorem 2.2 entails

$$\omega(-\mathbb{A})\le-\varpi+\zeta\ . \tag{2.12}$$

Let $\lambda\in\mathbb{C}$ be such that $\mathrm{Re}\,\lambda>-\varpi$ if $a_m=\infty$. Observe that the solution to

$$\partial_a\phi+A_\lambda(a)\phi=0\ ,\quad a\in(0,a_m)\ ,\qquad \phi(0)=\int_0^{a_m}b(a)\,\phi(a)\,\mathrm{d}a\ ,$$

is given by

$$\phi(a)=\Pi_\lambda(a,0)\phi(0)\ ,\quad a\in(0,a_m)\ ,\qquad \phi(0)=Q_\lambda\phi(0)\ ,$$

where

$$Q_\lambda:=\int_0^{a_m}b(a)\,\Pi_\lambda(a,0)\,\mathrm{d}a\ .$$

As we shall see, the spectrum of $-\mathbb{A}$ and thus the asymptotic behavior of solutions to (1.5)-(1.7) is determined by the spectral radii of the λ-dependent family Q_λ. From (2.3), (2.4), and (2.6) we deduce the regularizing property

$$Q_\lambda\in\mathcal{L}(E_0,E_\theta)\cap\mathcal{L}(E_{1-\theta},E_1)\ ,\qquad \theta\in[0,1)\ , \tag{2.13}$$

and hence $Q_\lambda|_{E_\theta}\in\mathcal{L}(E_\theta)$ is compact for $\theta\in[0,1)$ due to the compact embedding of E_α in E_β for $0\le\beta<\alpha<1$. Consequently, $\sigma(Q_\lambda|_{E_\theta})\setminus\{0\}$ consists only of eigenvalues.

Lemma 2.4. *Let $\lambda \in \mathbb{R}$ with $\lambda > -\varpi$ if $a_m = \infty$. Then the spectral radius $r(Q_\lambda)$ is positive and a simple eigenvalue of $Q_\lambda \in \mathcal{L}(E_0)$ with an eigenvector in E_1 that is quasi-interior in E_0^+. It is the only eigenvalue of Q_λ with a positive eigenvector. Moreover, $\sigma(Q_\lambda|_{E_\theta}) \setminus \{0\} = \sigma(Q_\lambda) \setminus \{0\}$ for $\theta \in [0,1)$.*

Proof. Since $Q_\lambda \in \mathcal{L}(E_0)$ is compact and irreducible according to (2.7) (see the proof of [19, Lem.2.1]), it is a classical result that the spectral radius $r(Q_\lambda)$ is positive and a simple eigenvalue of Q_λ with a quasi-interior eigenvector [3, Thm.12.3]. This eigenvector belongs to E_1 owing to (2.13). The regularizing property (2.3) also ensures the last statement. □

In view of (2.13) and the observations stated in Lemma 2.4 we shall not distinguish between $Q_\lambda \in \mathcal{L}(E_0)$ and $Q_\lambda|_{E_\theta} \in \mathcal{L}(E_\theta)$ in the sequel if $\theta \in [0,1)$.

The arguments used in the proof of [19, Lem.2.2] reveal:

Lemma 2.5. *Let $I = \mathbb{R}$ if $a_m < \infty$ and $I = (-\varpi, \infty)$ if $a_m = \infty$. Then the mapping*

$$[\lambda \mapsto r(Q_\lambda)] : I \to (0,\infty)$$

is continuous, strictly decreasing, and $\lim_{\lambda\to\infty} r(Q_\lambda) = 0$. If $a_m < \infty$, then $\lim_{\lambda\to-\infty} r(Q_\lambda) = \infty$.

Next, we characterize the resolvent of $-\mathbb{A}$.

Lemma 2.6. *Consider $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\,\lambda > -\varpi + \zeta$ and suppose that $1 - Q_\lambda \in \mathcal{L}(E_0)$ is bijective. Then*

$$\big[(\lambda+\mathbb{A})^{-1}\phi\big](a) = \int_0^a \Pi_\lambda(a,\sigma)\,\phi(\sigma)\,\mathrm{d}\sigma + \Pi_\lambda(a,0)(1-Q_\lambda)^{-1}\int_0^{a_m} b(s)\int_0^s \Pi_\lambda(s,\sigma)\,\phi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \tag{2.14}$$

for $a \in J$ and $\phi \in \mathbb{E}_0$.

Proof. By (2.12), any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\varpi + \zeta$ belongs to the resolvent set of $-\mathbb{A}$, so it follows from the Laplace transform formula and (2.8) that for $\phi \in \mathbb{E}_0$ and a.a. $a \in J$ we have

$$\big[(\lambda+\mathbb{A})^{-1}\phi\big](a) = \int_0^\infty e^{-\lambda t}\,\big[S(t)\,\phi\big](a)\,\mathrm{d}t = \int_0^a \Pi_\lambda(a,t)\,\phi(t)\,\mathrm{d}t + \Pi_\lambda(a,0)\int_0^\infty e^{-\lambda t} B_\phi(t)\,\mathrm{d}t\ .$$

Next, from (2.11),

$$\Psi := \int_0^\infty e^{-\lambda t} B_\phi(t)\,\mathrm{d}t \in E_0$$

and, on using (2.8) and (2.10), we obtain

$$\begin{aligned}\Psi &= \int_0^{a_m} b(a)\int_0^\infty e^{-\lambda t}\,\big[S(t)\,\phi\big](a)\,\mathrm{d}t\,\mathrm{d}a\\ &= \int_0^{a_m} b(a)\,\Pi_\lambda(a,0)\,\mathrm{d}a\,\Psi + \int_0^{a_m} b(a)\int_0^a \Pi_\lambda(a,t)\,\phi(t)\,\mathrm{d}t\,\mathrm{d}a\ ,\end{aligned}$$

that is,

$$\Psi = (1-Q_\lambda)^{-1}\int_0^{a_m} b(a)\int_0^a \Pi_\lambda(a,t)\,\phi(t)\,\mathrm{d}t\,\mathrm{d}a$$

from which the claim follows. □

Observe that Lemma 2.6 also holds without assumption (2.5) on maximal regularity of $-A$ and for $\phi \in L_1(J,E_0)$, i.e. for $p=1$. However, (2.5) allows us to interpret formula (2.14) in an adequate functional setting:

Remark 2.7. *Let* $\phi \in \mathbb{E}_0$, *let* $\lambda \in \mathbb{C}$ *be such that* $\mathrm{Re}\,\lambda > -\varpi + \zeta$, *and suppose that* $1 - Q_\lambda \in \mathcal{L}(E_0)$ *is bijective. Recall from Lemma 2.4 that* $(1 - Q_\lambda)^{-1} \in \mathcal{L}(E_\varsigma)$. *Then, by* (2.5),

$$(\lambda + \mathbb{A})^{-1}\phi = v_\lambda \phi + w_\lambda \phi \ , \tag{2.15}$$

where maximal regularity of A_λ *implies that* $v_\lambda \phi \in \mathbb{E}_1$, *given by*

$$(v_\lambda \phi)(a) := \int_0^a \Pi_\lambda(a,\sigma)\,\phi(\sigma)\ \mathrm{d}\sigma \ , \quad a \in J \ ,$$

is the unique solution to the Cauchy problem

$$\partial_a v + A_\lambda v = \phi \ , \quad a \in (0, a_m) \ , \qquad v(0) = 0 \ ,$$

and $w_\lambda \phi \in \mathbb{E}_1$, *given by*

$$(w_\lambda \phi)(a) := \Pi_\lambda(a,0)(1 - Q_\lambda)^{-1} \int_0^{a_m} b(s)\,(v_\lambda \phi)(s)\,\mathrm{d}s \ , \quad a \in J \ ,$$

is the unique solution to the Cauchy problem

$$\partial_a w + A_\lambda w = 0 \ , \quad a \in (0, a_m) \ , \qquad w(0) = (1 - Q_\lambda)^{-1} \int_0^{a_m} b(s)\,(v_\lambda \phi)(s)\,\mathrm{d}s \in E_\varsigma \ .$$

The characterization of the generator $-\mathbb{A}$ of the semigroup $\{S(t)\,;\, t \geq 0\}$ from Theorem 2.2 is now straightforward.

Theorem 2.8. $\phi \in \mathbb{E}_0$ *belongs to the domain* $\mathrm{dom}(-\mathbb{A})$ *of* $-\mathbb{A}$ *if and only if* $\phi \in \mathbb{E}_1$ *with*

$$\phi(0) = \int_0^{a_m} b(a)\,\phi(a)\ \mathrm{d}a \ . \tag{2.16}$$

Moreover, $\mathbb{A}\phi = \partial_a \phi + A\phi$ *for* $\phi \in \mathrm{dom}(-\mathbb{A})$.

Proof. By Lemma 2.5, we can choose $\lambda > -\varpi + \zeta$ such that $1 - Q_\lambda \in \mathcal{L}(E_0)$ is boundedly invertible. Thus λ belongs to the resolvent set of $-\mathbb{A}$ by Theorem 2.2. Remark 2.7 easily gives $\mathrm{dom}(-\mathbb{A}) \subset \mathbb{E}_1$. Moreover, if $\psi \in \mathbb{E}_0$ and $\phi := (\lambda + \mathbb{A})^{-1}\psi \in \mathrm{dom}(-\mathbb{A})$, then $\phi(0) \in E_\varsigma$ by (2.1) and

$$\phi(0) = (w_\lambda \psi)(0) = (1 - Q_\lambda)^{-1} \int_0^{a_m} b(a)\,(v_\lambda \psi)(a)\ \mathrm{d}a \ .$$

The same calculations as in the proof of Lemma 2.6 yield

$$\int_0^{a_m} b(a)\,\phi(a)\,\mathrm{d}a = \int_0^\infty e^{-\lambda t} \int_0^{a_m} b(a)\,[S(t)\psi](a)\,\mathrm{d}a\,\mathrm{d}t = (1 - Q_\lambda)^{-1} \int_0^{a_m} b(a)\,(v_\lambda \psi)(a)\,\mathrm{d}a = \phi(0) \ .$$

Conversely, if $\phi \in \mathbb{E}_1$ satisfies (2.16), then $\psi := (\partial_a + A_\lambda)\phi \in \mathbb{E}_0$ by (2.2) and, since $\phi(t) \in E_1$ for a.a. $t \in J$, we have

$$\frac{\partial}{\partial t}\big(\Pi_\lambda(a,t)\phi(t)\big) = \Pi_\lambda(a,t)\psi(t)$$

for a.a. $t \in J$ and $a > t$ due to the fact that Π_λ is the parabolic evolution operator for A_λ. Integration with respect to t gives

$$(v_\lambda \psi)(a) = \phi(a) - \Pi_\lambda(a,0)\phi(0)$$

from which we obtain

$$(w_\lambda \psi)(a) = \Pi_\lambda(a,0)(1 - Q_\lambda)^{-1} \int_0^{a_m} b(s)\,\big[\phi(s) - \Pi_\lambda(s,0)\phi(0)\big]\,\mathrm{d}s = \Pi_\lambda(a,0)\,\phi(0)$$

for $a \in J$, whence

$$\phi = v_\lambda \psi + w_\lambda \psi = (\lambda + \mathbb{A})^{-1}\psi \in \mathrm{dom}(-\mathbb{A}) \ .$$

Finally, $(\lambda + \mathbb{A})\phi = \psi = (\partial_a + A_\lambda)\phi$ and the proof is complete. □

Remark 2.9. *Theorem 2.2 and Theorem 2.8 show that for any initial value $\phi \in \mathbb{E}_1$ satisfying* (2.16)*, the unique solution $u \in C(\mathbb{R}^+, \mathbb{E}_1) \cap C^1(\mathbb{R}^+, \mathbb{E}_0)$ to* (1.5)-(1.7) *is given by $u(t) = S(t)\phi$, $t \geq 0$, with $S(t)$ defined in* (2.8)*. If ϕ is only in $\mathbb{E}_0$, then $u(t) = S(t)\phi$, $t \geq 0$, defines a mild solution in $C(\mathbb{R}^+, \mathbb{E}_0)$. Moreover, $u(t) \in \mathbb{E}_0^+$ for $t \geq 0$ if $\phi \in \mathbb{E}_0^+$.*

The characterization of the generator in Theorem 2.8 allows us also to consider (momentarily) time-dependent operators $\mathbb{A} = \mathbb{A}(t)$ with time-independent domains by applying the theory for hyperbolic evolution systems, e.g. see [9, 10]. More precisely, we have:

Proposition 2.10. *Let b satisfy* (2.6)*. Let $T > 0$ and consider*

$$A \in C^1\big([0,T], L_\infty(J, \mathcal{L}(E_1, E_0))\big)$$

such that $A(t,\cdot)$ satisfies for each $t \in [0,T]$ assumptions (2.2), (2.3), (2.4), (2.5), *and* (2.7)*, where σ, κ, ν, and ϖ therein may depend on t. Further suppose that*

$$\|\Pi_{A(t,\cdot)}(a,r)\|_{\mathcal{L}(E_0)} \leq e^{\eta(a-r)} \ , \quad 0 \leq r \leq a < a_m \ , \tag{2.17}$$

for some $\eta \in \mathbb{R}$ being independent of $t \in [0,T]$. For $t \in [0,T]$ define $\mathbb{A}(t) := \partial_a + A(t,\cdot)$ with domain given by $\mathbb{D} := \{\phi \in \mathbb{E}_1\, ;\, \phi$ satisfies (2.16)$\}$*. Then there is a unique, positive evolution system $\mathbb{U}(t,s)$, $0 \leq s \leq t \leq T$ in $\mathbb{E}_0$ to*

$$\dot{u} + \mathbb{A}(t)u = 0 \ , \quad t \in (s,T] \ , \quad u(s) = \phi$$

satisfying $(E_1) - (E_5)$ in [10, Chapt.5]*. In particular, this problem has for each $\phi \in \mathbb{D}$ a unique solution $u \in C^1\big((s,T], \mathbb{E}_0\big) \cap C\big([s,T], \mathbb{D}\big)$.*

Proof. It follows from Theorem 2.2 and Theorem 2.8 together with Remark 2.3 and (2.17) that $\{-\mathbb{A}(t)\, ;\, t \in [0,T]\}$ is in the sense of [10, Chapt.5] a stable family of infinitesimal generators of strongly continuous semigroups on $\mathbb{E}_0$ with time-independent domain $\mathbb{D}$. Moreover,

$$\big(t \mapsto \mathbb{A}(t)\phi\big) \in C^1\big([0,T], \mathbb{E}_0\big)$$

for $\phi \in \mathbb{D}$. The assertion is now a consequence of [9] (see [10, 5.Thm.4.8]). $\square$

3. Stability of the Trivial Solution and Asynchronous Exponential Growth

We now restrict our attention again to the time-independent situation and characterize the growth bound $\omega(-\mathbb{A})$ of $-\mathbb{A}$. We first analyze the point spectrum of $-\mathbb{A}$ and extend formula (2.14) to a larger class of λ values.

Lemma 3.1. *(i) Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\varpi$ if $a_m = \infty$ and let $m \in \mathbb{N} \setminus \{0\}$. Then $\lambda \in \sigma_p(-\mathbb{A})$ with geometric multiplicity m if and only if $1 \in \sigma_p(Q_\lambda)$ with geometric multiplicity m.*
(ii) Formula (2.14) *holds for any $\lambda \in \varrho(-\mathbb{A})$ provided $\mathrm{Re}\,\lambda > -\varpi$ if $a_m = \infty$.*

Proof. (i) Let $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > -\varpi$ if $a_m = \infty$. Suppose $\lambda \in \sigma_p(-\mathbb{A})$ has geometric multiplicity m so that there are linearly independent $\phi_1, ..., \phi_m \in \mathrm{dom}(-\mathbb{A})$ with $(\lambda + \mathbb{A})\phi_j = 0$ for $j = 1, ..., m$. From Theorem 2.8 we deduce

$$\phi_j(a) = \Pi_\lambda(a,0)\phi_j(0) \qquad \text{with} \qquad \phi_j(0) = Q_\lambda \phi_j(0) \ .$$

Hence, $\phi_1(0), ...\phi_m(0)$ are necessarily linearly independent eigenvectors of Q_λ corresponding to the eigenvalue 1. Now, suppose $1 \in \sigma_p(Q_\lambda)$ has geometric multiplicity m so that there are linearly independent $\Phi_1, ...\Phi_m \in E_\varsigma$ with $Q_\lambda \Phi_j = \Phi_j$ for $j = 1, ..., m$. Put $\phi_j := \Pi_\lambda(\cdot, 0)\Phi_j \in \mathbb{E}_1$ and note that, for $j = 1, ..., m$,

$$\partial_a \phi_j + A_\lambda \phi_j = 0 \ , \quad \int_0^{a_m} b(a)\, \phi_j(a)\, \mathrm{d}a = Q_\lambda \Phi_j = \Phi_j = \phi_j(0) \ .$$

Thus $\phi_j \in \mathrm{dom}(-\mathbb{A})$ and $(\lambda+\mathbb{A})\phi_j = 0$ by Theorem 2.8, i.e. $\lambda \in \sigma_p(-\mathbb{A})$. If $\alpha_1, ..., \alpha_m$ are any scalars, the unique solvability of the Cauchy problem

$$\partial_a \phi + A_\lambda \phi = 0\ , \quad a \in (0, a_m)\ , \qquad \phi(0) = \sum_j \alpha_j\, \Phi_j$$

ensures that $\phi_1, ..., \phi_m$ are linearly independent. This proves (i).
(ii) Let $\lambda \in \varrho(-\mathbb{A})$ with $\mathrm{Re}\,\lambda > -\varpi$ if $a_m = \infty$. Then A_λ has maximal regularity due to (2.5) and, by Theorem 2.8, $(\lambda + \mathbb{A})\psi = \phi$ with $\phi \in \mathbb{E}_0$ and $\psi \in \mathbb{E}_1$ if and only if $(\partial_a + A_\lambda)\psi = \phi$ with

$$\psi(0) = \int_0^{a_m} b(a)\, \psi(a)\; \mathrm{d}a\ ,$$

that is,

$$\psi(a) = \Pi_\lambda(a,0)\psi(0) + \int_0^a \Pi_\lambda(a,\sigma)\, \phi(\sigma)\; \mathrm{d}\sigma\ ,$$

$$(1 - Q_\lambda)\psi(0) = \int_0^{a_m} b(a) \int_0^a \Pi_\lambda(a,\sigma)\, \phi(\sigma)\; \mathrm{d}\sigma\, \mathrm{d}a\ .$$

Since $\lambda \in \varrho(-\mathbb{A})$ and since Q_λ is compact, (i) ensures that $1 \in \varrho(Q_\lambda)$, hence $1 - Q_\lambda$ is invertible and so

$$\psi(0) = (1 - Q_\lambda)^{-1} \int_0^{a_m} b(a) \int_0^a \Pi_\lambda(a,\sigma)\, \phi(\sigma)\; \mathrm{d}\sigma\, \mathrm{d}a\ .$$

As $\psi = (\lambda + \mathbb{A})^{-1}\phi$, this gives formula (2.14). □

Recall that the α-growth bound $\omega_1(-\mathbb{A})$ of $-\mathbb{A}$ is defined by

$$\omega_1(-\mathbb{A}) := \lim_{t\to\infty} \frac{1}{t} \log\big(\alpha(S(t))\big)\ ,$$

where α denotes Kuratowski's measure of non-compactness. That is, if B is a subset of a normed vector space X, then $\alpha(B)$ is defined as the infimum over all $\delta > 0$ such that B can be covered with finitely many sets of diameter less than δ, and if T is a bounded operator on X, then $\alpha(T)$ is the infimum over all $\varepsilon > 0$ such that $\alpha(T(B)) \le \varepsilon\alpha(B)$ for any bounded set $B \subset X$. Recall that $\omega_1(-\mathbb{A}) \le \omega(-\mathbb{A})$.

We next provide bounds on $\omega_1(-\mathbb{A})$.

Lemma 3.2. *There holds*

$$\sup\{\mathrm{Re}\,\lambda\,;\, \lambda \in \sigma_e(-\mathbb{A})\} \;\le\; \omega_1(-\mathbb{A}) \;\le\; -\varpi\ .$$

Moreover, if $a_m < \infty$, then $\omega_1(-\mathbb{A}) = -\infty$ and the semigroup $\{S(t)\,;\, t \ge 0\}$ is eventually compact.

Proof. The first inequality of the assertion is generally true for strongly continuous semigroups [22, Prop.4.13]. We thus merely have to show that $\omega_1(-\mathbb{A}) \le -\varpi$ which can be done along the lines of the scalar case [22, Thm.4.6]: Let $t > 0$ and write $S(t) = U(t) + W(t)$, where $U(t), W(t) \in \mathcal{L}(\mathbb{E}_0)$ are defined as

$$\big[U(t)\,\phi\big](a) := \begin{cases} 0\ , & a \in (0,t)\ , \\ \big[S(t)\,\phi\big](a)\ , & a \in (t, a_m)\ , \end{cases} \qquad \big[W(t)\,\phi\big](a) := \begin{cases} \big[S(t)\,\phi\big](a)\ , & a \in (0,t)\ , \\ 0\ , & a \in (t, a_m)\ , \end{cases}$$

for $a \in J$, $\phi \in \mathbb{E}_0$. Observing that $\alpha(S(t)) \le \alpha(U(t)) + \alpha(W(t))$ and, by (2.3) and (2.8),

$$\alpha(U(t)) \le \|U(t)\|_{\mathcal{L}(\mathbb{E}_0)} \le Me^{-\varpi t}\ , \quad t < a_m\ , \tag{3.1}$$

the assertion follows from the definition of $\omega_1(-\mathbb{A})$ provided we can show that $\alpha(W(t)) = 0$. For this it suffices to show that if B is any bounded subset of $\mathbb{E}_0$, then $W(t)B$ is relatively compact in $\mathbb{E}_0$. We use Kolmogorov's compactness criterion [6, Thm.A.1] for which we may assume without loss of generality that $a_m=\infty$. Clearly, Theorem 2.2 ensures that $W(t)B$ is bounded in $\mathbb{E}_0$. If $\phi\in\mathbb{E}_0$ and $h>0$, then

$$\begin{aligned}\int_0^\infty &\|[W(t)\phi](a+h)-[W(t)\phi](a)\|_{E_0}^p\,\mathrm{d}a\\ &\le\int_0^{t-h}\|\Pi(a+h,0)-\Pi(a,0)\|^p_{\mathcal{L}(E_\xi,E_0)}\,\|B_\phi(t-a-h)\|^p_{E_\xi}\,\mathrm{d}a\\ &\quad+\int_0^{t-h}\|\Pi(a,0)\|^p_{\mathcal{L}(E_0)}\,\|B_\phi(t-a-h)-B_\phi(t-a)\|^p_{E_0}\,\mathrm{d}a\\ &\quad+\int_{t-h}^{t}\|\Pi(a,0)\|^p_{\mathcal{L}(E_0)}\,\|B_\phi(t-a)\|^p_{E_0}\,\mathrm{d}a\ ,\end{aligned}$$

where $\xi\in(0,1/p)$. On using (2.3) and Lemma 2.1 it is readily seen that the second and third integral on the right side tend to zero as $h\to 0$, uniformly with respect to $\phi\in B$. For the first integral we use the fact (see [1, II.Eq.(5.3.8)]) that

$$\|\Pi(a+h,0)-\Pi(a,0)\|_{\mathcal{L}(E_\xi,E_0)}\le c(t)h^\xi\ ,\quad a\le t\ ,$$

to obtain from Lemma 2.1 the estimate

$$\int_0^{t-h}\|\Pi(a+h,0)-\Pi(a,0)\|^p_{\mathcal{L}(E_\xi,E_0)}\,\|B_\phi(t-a-h)\|^p_{E_\xi}\,\mathrm{d}a\le c(t)^p h^{\xi p}\int_0^{t-h}(t-a-h)^{-\xi p}\,\mathrm{d}a\,\|\phi\|^p_{\mathbb{E}_0}$$

with right hand side tending to zero as $h\to 0$, uniformly with respect to $\phi\in B$. Next, by (2.3), (2.8), and Lemma 2.1,

$$\|[W(t)\phi](a)\|_{E_\xi}\le c(B,t)\,(t-a)^{-\xi}\ ,\quad a<t\ ,$$

for some constant $c(B,t)$. Given $\varepsilon\in(0,t)$ let R_ε be the E_0-closure of the ball in E_ξ centered at 0 of radius $c(B,t)\varepsilon^{-\xi}$. Then R_ε is compact in E_0 since E_ξ embeds compactly in E_0 and

$$[W(t)\phi](a)\in R_\varepsilon\ ,\quad a\in\mathbb{R}^+\setminus[t-\varepsilon,t]\ ,\quad \phi\in B\ .$$

Therefore, [6, Thm.A.1] implies that $W(t)B$ is relatively compact in $\mathbb{E}_0$, hence $\omega_1(-\mathbb{A})\le-\varpi$. Finally, if $a_m<\infty$ and $t>a_m$, then $U(t)=0$ and so $S(t)=W(t)$. This proves the lemma. □

Lemma 3.3. *Let $\lambda\in\sigma(-\mathbb{A})$ with $\mathrm{Re}\,\lambda>-\varpi$ if $a_m=\infty$. Then $\lambda\in\sigma_p(-\mathbb{A})\setminus\sigma_e(-\mathbb{A})$ and $1\in\sigma_p(Q_\lambda)$. Moreover, λ is isolated in $\sigma(-\mathbb{A})$ and a pole of the resolvent $[\tau\mapsto(\tau+\mathbb{A})^{-1}]$. The residue of the resolvent at λ,*

$$P_\lambda:=\frac{1}{2\pi i}\int_\Gamma(\tau+\mathbb{A})^{-1}\,\mathrm{d}\tau\ ,$$

is a projection on $\mathbb{E}_0$ and $\mathbb{E}_0=\mathrm{im}(P_\lambda)\oplus\mathrm{im}(1-P_\lambda)$ with $\mathrm{im}(P_\lambda)=\ker(\lambda+\mathbb{A})^m$, where Γ is a positively oriented closed curve in the complex plane such that no point in $\sigma(-\mathbb{A})$ lies in or on Γ and $m\in\mathbb{N}$ is the order of the pole λ.

Proof. Let $\lambda\in\sigma(-\mathbb{A})$ with $\mathrm{Re}\,\lambda>-\varpi$ if $a_m=\infty$. Since $-\mathbb{A}$ is closed and $\lambda\in\sigma(-\mathbb{A})\setminus\sigma_e(-\mathbb{A})$ by Lemma 3.2, it follows from [2] that $\lambda\in\sigma_p(-\mathbb{A})$ is isolated in $\sigma(-\mathbb{A})$ and a pole of the resolvent $[\tau\mapsto(\tau+\mathbb{A})^{-1}]$. In particular, $\lambda\in\sigma_p(-\mathbb{A})$ implies $1\in\sigma_p(Q_\lambda)$ by Lemma 3.1(i). Since λ is isolated in $\sigma(-\mathbb{A})$, the remaining assertions follow from Laurent series theory described in [8, 23], for details we refer to [20, Prop.4.8, Prop.4.11]. □

Let us recall from [21, Prop.2.5] that the peripheral spectrum of the generator of a strongly continuous positive semigroup in a Banach lattice consists exactly of the generator's spectral bound provided the latter is strictly greater than the α-growth bound. The proof of this result is based on [4]. As this turns out to be a crucial tool for our purpose, we state it explicitly:

Proposition 3.4. *If $s(-\mathbb{A})>\omega_1(-\mathbb{A})$, then $\sigma_0(-\mathbb{A})=\{s(-\mathbb{A})\}$.*

We now give a criterion for the spectral bound to be negative. Note that this result implies the global asymptotic stability of the trivial solution to (1.5)-(1.6).

Theorem 3.5. *If $r(Q_0) < 1$, then $\omega(-\mathbb{A}) < 0$.*

Proof. Let $\hat\lambda_0 := s(-\mathbb{A})$. Since $r(Q_0) < 1$ there is $\delta > 0$ such that $r(Q_{-\delta}) < 1$ by Lemma 2.5. Fix $\varepsilon \in (0, \delta)$ with $\varepsilon < \varpi$ if $a_m = \infty$, see (2.4). Suppose there is $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq -\varepsilon$ and $1 \in \sigma_p(Q_\lambda)$. Then $\lambda \in \sigma(-\mathbb{A})$ due to Lemma 3.1, whence $\hat\lambda_0 \geq -\varepsilon$. Since $\varepsilon < \varpi$ if $a_m = \infty$, Lemma 3.2 and Proposition 3.4 entail $\sigma_0(-\mathbb{A}) = \{\hat\lambda_0\}$. Invoking Lemma 3.1 again we see that $1 \in \sigma_p(Q_{\hat\lambda_0})$ and so $r(Q_{\hat\lambda_0}) \geq 1$. Lemma 2.5 then gives $\hat\lambda_0 < -\delta$ contradicting $\hat\lambda_0 \geq -\varepsilon$ and $\varepsilon < \delta$. Consequently,

$$\sup\big\{\mathrm{Re}\,\lambda\,;\, 1 \in \sigma_p(Q_\lambda)\big\} \leq -\varepsilon < 0\ . \tag{3.2}$$

Recall e.g. from [20, Prop.4.13] that

$$\omega(-\mathbb{A}) = \max\big\{\omega_1(-\mathbb{A}), \sup_{\lambda\in\sigma(-\mathbb{A})\setminus\sigma_e(-\mathbb{A})} \mathrm{Re}\,\lambda\big\}\ .$$

Since $\varpi > 0$ if $a_m = \infty$, the assertion is now an immediate consequence of (3.2), Lemma 3.2, and Lemma 3.3. □

Next, we provide a criterion for asynchronous exponential growth of the semigroup $\{S(t); t \geq 0\}$ which is analogous to the scalar case $A \equiv \mu$ in [20]. For similar results in the spatially inhomogeneous setting we refer to [14, 12]. We first need an auxiliary result.

Lemma 3.6. *Let $\lambda_0 \in \mathbb{R}$ with $\lambda_0 > -\varpi$ if $a_m = \infty$ be such that $r(Q_{\lambda_0}) = 1$. Then λ_0 is a simple eigenvalue of $-\mathbb{A}$.*

Proof. According to Lemma 2.4 there is a quasi-interior eigenvector $\Phi_0 \in E_1$ of Q_{λ_0} corresponding to the simple eigenvalue $r(Q_{\lambda_0}) = 1$. By Theorem 2.8 and Lemma 3.1, $\ker(\lambda_0 + \mathbb{A})$ is one-dimensional and spanned by $\varphi := \Pi_{\lambda_0}(\cdot, 0)\Phi_0$. It thus remains to show that $\ker(\lambda_0 + \mathbb{A})^2 \subset \ker(\lambda_0 + \mathbb{A})$. Let $\psi \in \ker(\lambda_0 + \mathbb{A})^2$ and set

$$\phi := (\lambda_0 + \mathbb{A})\psi \in \ker(\lambda_0 + \mathbb{A})\ .$$

Then $\phi = \xi\varphi$ for some $\xi \in \mathbb{R}$. Suppose $\xi \neq 0$, so without loss of generality $\xi > 0$. Let $\tau > 0$ be such that $\tau\Phi_0 + \psi(0) \in E_\varsigma^+ \setminus \{0\}$ and put $q := \tau\varphi + \psi \in \mathrm{dom}(-\mathbb{A})$. Then $(\lambda_0 + \mathbb{A})q = \phi$ and from Theorem 2.8 it follows that

$$q(a) = \Pi_{\lambda_0}(a, 0)q(0) + \xi \int_0^a \Pi_{\lambda_0}(a, \sigma)\, \Pi_{\lambda_0}(\sigma, 0)\, \Phi_0\ \mathrm{d}\sigma = \Pi_{\lambda_0}(a, 0)q(0) + a\,\xi\, \Pi_{\lambda_0}(a, 0)\, \Phi_0$$

and

$$q(0) = \int_0^{a_m} b(a)\, q(a)\ \mathrm{d}a\ .$$

Plugging the former into the second formula yields

$$(1 - Q_{\lambda_0})q(0) = \xi \int_0^{a_m} b(a)\, a\, \Pi_{\lambda_0}(a, 0)\, \Phi_0\ \mathrm{d}a\ .$$

As $q(0)$ and the right hand side are both positive and nonzero, we derive from [3, Cor.12.4] a contradiction to $r(Q_{\lambda_0}) = 1$. Consequently, $\xi = 0$ and the claim follows because now $\phi = 0$. □

Exponential asynchronous growth of the semigroup $\{S(t); t \geq 0\}$ given in (2.8) is now an easy consequence of [21]. Recall from Lemma 3.3 that $P_{\lambda_0} : \mathbb{E}_0 \to \ker(\lambda_0 + \mathbb{A})$ is a projection with rank one.

Theorem 3.7. *Suppose that $r(Q_\alpha) > 1$ for some $\alpha \in \mathbb{R}$ with $\alpha > -\varpi$ if $a_m = \infty$. Then $\{S(t)\,;\,t \ge 0\}$ has asynchronous exponential growth with intrinsic growth constant λ_0, that is,*

$$e^{-\lambda_0 t}\,S(t) \longrightarrow P_{\lambda_0} \quad in \quad \mathcal{L}(\mathbb{E}_0) \quad as \quad t \to \infty\ ,$$

where $\lambda_0 > \alpha$ is the unique number satisfying $r(Q_{\lambda_0}) = 1$.

Proof. By Lemma 2.5 there is a unique $\lambda_0 > \alpha$ such that $r(Q_{\lambda_0}) = 1$. Let $\hat\lambda_0 := s(-\mathbb{A})$ denote the spectral bound of $-\mathbb{A}$. Then, owing to Lemma 2.4 and Lemma 3.3, $\hat\lambda_0 \ge \lambda_0$ and so $\hat\lambda_0 > \omega_1(-\mathbb{A})$ according to Lemma 3.2 since $\lambda_0 > -\varpi$ if $a_m = \infty$ or $\omega_1(-\mathbb{A}) = -\infty$ if $a_m < \infty$. Thus, $\sigma_0(-\mathbb{A}) = \{\hat\lambda_0\}$ by Proposition 3.4, and then $1 \in \sigma_p(Q_{\hat\lambda_0})$ by Lemma 3.3 from which $1 \le r(Q_{\hat\lambda_0})$. However, due to $\hat\lambda_0 \ge \lambda_0$ and Lemma 2.5 we have $r(Q_{\hat\lambda_0}) \le r(Q_{\lambda_0}) = 1$, whence $\hat\lambda_0 = \lambda_0$. Consequently, $\sigma_0(-\mathbb{A}) = \{\lambda_0\}$ and $\lambda_0 > \omega_1(-\mathbb{A})$. Finally, Lemma 3.6 together with Lemma 3.3 imply that λ_0 is a simple pole of the resolvent $(\tau + \mathbb{A})^{-1}$. The assertion then follows from [21, Prop.2.3]. □

To derive a formula for the projection $P_{\lambda_0} : \mathbb{E}_0 \to \ker(\lambda_0 + \mathbb{A})$ from Theorem 3.7 observe that there is a quasi-interior element Φ_0 in E_1 such that $\ker(1 - Q_{\lambda_0}) = \mathrm{span}\{\Phi_0\}$ and $\ker(\lambda_0 + \mathbb{A}) = \mathrm{span}\{\Pi_{\lambda_0}(\cdot, 0)\Phi_0\}$. Let $\phi \in \mathbb{E}_0$ be fixed and let $c(\phi) \in \mathbb{R}$ be such that $P_{\lambda_0}\phi = c(\phi)\Pi_{\lambda_0}(\cdot, 0)\Phi_0$. Recall that λ_0 is a simple pole of the resolvent $(\tau + \mathbb{A})^{-1}$. Since v_λ is holomorphic in λ, it follows from (2.15) and the Residue Theorem that

$$P_{\lambda_0}\phi = \lim_{\lambda\to\lambda_0} (\lambda - \lambda_0)\Pi_\lambda(\cdot, 0)(1 - Q_\lambda)^{-1}H_\lambda\phi\ , \qquad H_\lambda\phi := \int_0^{a_m} b(s) \int_0^s \Pi_\lambda(s, \sigma)\,\phi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s\ .$$

Let $w' \in E_0'$ be a positive eigenfunctional of the dual operator Q'_{λ_0} of Q_{λ_0} corresponding to the eigenvalue $r(Q_{\lambda_0}) = 1$. Then, for $f' \in \mathbb{E}_0'$ defined by

$$\langle f', \psi\rangle := \langle w', \int_0^{a_m} b(a)\psi(a)\mathrm{d}a\rangle\ , \quad \psi \in \mathbb{E}_0\ ,$$

we have, due to $Q'_{\lambda_0}w' = w'$, that

$$\begin{aligned} c(\phi)\langle w', \Phi_0\rangle = \langle f', P_{\lambda_0}\phi\rangle &= \lim_{\lambda\to\lambda_0} \langle f', (\lambda - \lambda_0)\Pi_\lambda(\cdot, 0)(1 - Q_\lambda)^{-1}H_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_0} \langle w', (\lambda - \lambda_0)(Q_\lambda - 1 + 1)(1 - Q_\lambda)^{-1}H_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_0} \langle w', (\lambda - \lambda_0)(1 - Q_\lambda)^{-1}H_\lambda\phi\rangle\ . \end{aligned}$$

Writing

$$H_\lambda\phi = d\big(H_\lambda\phi\big)\Phi_0 \oplus (1 - Q_{\lambda_0})g\big(H_\lambda\phi\big) \tag{3.3}$$

according to the decomposition $E_0 = \mathbb{R}\cdot\Phi_0 \oplus \mathrm{rg}(1 - Q_{\lambda_0})$, it follows

$$\lim_{\lambda\to\lambda_0} \langle w', (\lambda - \lambda_0)(1 - Q_\lambda)^{-1}H_\lambda\phi\rangle = d\big(H_{\lambda_0}\phi\big) \lim_{\lambda\to\lambda_0} \langle w', (\lambda - \lambda_0)(1 - Q_\lambda)^{-1}\Phi_0\rangle$$

since Q_λ is continuous in λ. But, from (3.3),

$$\langle w', H_{\lambda_0}\phi\rangle = d\big(H_{\lambda_0}\phi\big)\,\langle w', \Phi_0\rangle$$

since $Q'_{\lambda_0}w' = w'$, whence $d\big(H_{\lambda_0}\phi\big) = \xi\langle w', H_{\lambda_0}\phi\rangle$ with $\xi^{-1} = \langle w', \Phi_0\rangle$. Similarly, decomposing

$$Z_\lambda := (\lambda - \lambda_0)(1 - Q_\lambda)^{-1}\Phi_0$$

we find

$$\lim_{\lambda\to\lambda_0} \langle w', Z_\lambda\rangle = \big(\lim_{\lambda\to\lambda_0} d\big(Z_\lambda\big)\big)\langle w', \Phi_0\rangle.$$

Gathering these observations, we derive

$$c(\phi)\,\langle w', \Phi_0\rangle = C_0\,\langle w', H_{\lambda_0}\phi\rangle\,\langle w', \Phi_0\rangle$$

for some number C_0. Consequently,

$$P_{\lambda_0}\phi = C_0 \langle w', H_{\lambda_0}\phi\rangle \Pi_{\lambda_0}(\cdot,0)\Phi_0 \ .$$

Since P_{λ_0} is a projection, i.e. $P_{\lambda_0}^2 = P_{\lambda_0}$, the constant C_0 is easily computed and we obtain:

Proposition 3.8. *Under the assumptions of Theorem 3.7, the projection P_{λ_0} is given by*

$$P_{\lambda_0}\phi = \frac{\langle w', H_{\lambda_0}\phi\rangle}{\langle w', \int_0^{a_m} ab(a)\Pi_{\lambda_0}(a,0)\mathrm{d}a\, \Phi_0\rangle}\Pi_{\lambda_0}(\cdot,0)\Phi_0 \tag{3.4}$$

for $\phi \in \mathbb{E}_0$, where

$$H_{\lambda_0}\phi = \int_0^{a_m} b(s) \int_0^s \Pi_{\lambda_0}(s,\sigma)\,\phi(\sigma)\ \mathrm{d}\sigma\,\mathrm{d}s$$

and $w' \in E_0'$ is a positive eigenfunctional to the dual operator Q_{λ_0}' of Q_{λ_0} corresponding to the eigenvalue $r(Q_{\lambda_0}) = 1$.

Situations in which assumptions (2.2)-(2.7) are satisfied occur, for instance, when A is a second order elliptic operator in divergence form. In particular, in example (1.1)-(1.4) from the introduction assumptions (2.2)-(2.7) are satisfied provided $a_m < \infty$ and the functions μ, b are (sufficiently) smooth, nonnegative, and bounded while d is smooth and strictly positive on $\bar{J}\times\bar{\Omega}$. If $a_m = \infty$ then one additionally requires for maximal regularity that the limits of $d(a,x)$ and its first order derivatives with respect to x exist as $a \to \infty$, uniformly in x (see [13, Thm.1.4]). We refer to [17, Sect.3], [19, Sect.3] for details and other concrete examples. Therein one also finds simple settings in which the spectral radii $r(Q_\lambda)$ can be computed easily which allows one to apply Theorem 3.5 and Theorem 3.7.

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