# GLOBAL CONTINUA OF POSITIVE SOLUTIONS FOR SOME QUASILINEAR PARABOLIC EQUATION WITH A NONLOCAL INITIAL CONDITION 

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#### Abstract

This paper is concerned with a quasilinear parabolic equation including a nonlinear nonlocal initial condition. The problem arises as equilibrium equation in population dynamics with nonlinear diffusion. We make use of global bifurcation theory to prove existence of an unbounded continuum of positive solutions.


## 1. Introduction

Age-structured models have a long history (see [24] and the references therein), both for the case without and with spatial movement of individuals. To give a prototypical example for the evolution of an age-structured population with quasilinear diffusion, let $u=u(t, a, x)$ denote the distribution density of individuals at time $t \geq 0$, spatial position $x \in \Omega \subset \mathbb{R}^{n}$, and age $a \in J:=$ $\left[0, a_{m}\right)$, where $a_{m} \in(0, \infty]$ denotes the maximal age. Suppose that individuals move within the space domain $\Omega$ and that dispersal speed $d>0$ depends on the local overall population; that is, suppose that movement of individuals is described by a density-dependent diffusion term $-\operatorname{div}_{x}\left(d(U, a, x) \nabla_{x} u\right)$, where

$$
\begin{equation*}
U(t, x):=\int_{0}^{a_{m}} u(t, a, x) \mathrm{d} a \tag{1.1}
\end{equation*}
$$

is the overall population at spatial position $x$ at time $t$. Assume further that individuals cannot leave the space region $\Omega$ so that the behavior on the boundary $\partial \Omega$ is given by a Neumann condition $\partial_{\nu} u=0$, with $v$ denoting the outward unit normal to $\partial \Omega$. Let $\mu=\mu(U, a, x)$ and $\beta(U, a, x)$ denote the death and birth modulus, respectively. Then the evolution of the population is governed by the equations

$$
\begin{array}{ll}
\partial_{t} u+\partial_{a} u-\operatorname{div}_{x}\left(d(U, a, x) \nabla_{x} u\right)+\mu(U, a, x) u=0, & t>0, a \in\left(0, a_{m}\right), x \in \Omega, \\
u(t, 0, x)=\int_{0}^{a_{m}} \beta(U, a, x) u(t, a, x) \mathrm{d} a, & t>0, \quad x \in \Omega, \\
\partial_{v} u(t, a, x)=0, & t>0, a \in\left(0, a_{m}\right), x \in \partial \Omega, \tag{1.4}
\end{array}
$$

subject to an initial condition at $t=0$. Treatment of well-posedness issues for equations of the form (1.1)-(1.4) can be found e.g. in [24, 20]. Understanding the asymptotic behavior of the evolution of structured populations demands precise information about equilibrium (i.e. timeindependent) solutions. The aim of this paper is to provide an abstract approach to establish equilibrium solutions, which then applies to rather general nonlinear equations modeling agestructured populations with quasilinear diffusion and so is not restricted to the particular example above. We thus may want to put (1.1)-(1.4) in a more abstract framework. Given a (sufficiently smooth) function $u$, set $U:=\int_{0}^{a_{m}} u(a, \cdot) \mathrm{d} a$ and introduce a linear operator $A(u, a): E_{1} \rightarrow E_{0}$ by

$$
A(u, a) v:=-\operatorname{div}_{x}\left(d(U, a, \cdot) \nabla_{x} u\right), \quad v \in E_{1}:=\left\{v \in W_{p}^{2}(\Omega) ; \partial_{v} v=0 \text { on } \partial \Omega\right\},
$$

[^0]where $E_{0}:=L_{p}(\Omega)$ with $p \in[1, \infty)$. In this setting, equilibrium solutions $u: J \rightarrow E_{1}$ to (1.1)-(1.4) satisfy
\[

$$
\begin{array}{ll}
\partial_{a} u+A(u, a) u+\mu(u, a) u=0, & t>0, \quad a \in J \backslash\{0\} \\
u(0)=\int_{0}^{a_{m}} \beta(u, a) u(a) \mathrm{d} a, & t>0 \tag{1.6}
\end{array}
$$
\]

where death rate $\mu$ and birth rate $\beta$ depend in some way on $u$ (e.g., on $U$ ). Clearly, $u \equiv 0$ is a solution to (1.5)-(1.6) which therefore has to be singled out in the further analysis. Moreover, since $u$ represents a density, solutions should be nonnegative, that is, should belong to the positive cone $E_{0}^{+}$at any age $a$.
For non-diffusive age-structured populations (i.e. $A \equiv 0$ in (1.5)), positive equilibria were established e.g. by means of fixed point theorems in conical shells [12, 23]. These results were carried over to the diffusive case as well [19]. However, the fact that $u \equiv 0$ is a solution to (1.5), (1.6) allows one also to interpret the problem of finding positive solutions as a bifurcation problem [7] and so to obtain more insight into the structure of the equilibria set. For this we write the birth modulus in the form $\beta(u, a)=\lambda b(u, a)$ and introduce in this way a bifurcation parameter $\lambda>0$ which determines the intensity of the individual's fertility while the qualitative structure of the fertility is modeled by the function $b$. Writing $\mathbb{A}(u):=A(u, \cdot)+\mu(u, \cdot)$ and

$$
\lambda \ell(v) u:=\int_{0}^{a_{m}} \lambda b(v, a) u(a) \mathrm{d} a
$$

we are concerned in this paper with finding values $\lambda>0$ and positive nontrivial solutions $u: J \rightarrow E_{0}^{+} \cap E_{1}$ to the nonlinear problem

$$
\begin{align*}
& \partial_{a} u+\mathbb{A}(u) u=0, \quad a \in J \backslash\{0\},  \tag{1.7}\\
& u(0)=\lambda \ell(u) u . \tag{1.8}
\end{align*}
$$

The results derived herein extend our previous results [18, 19] on local and global bifurcation. In [18] it was shown that if $\mathbb{A}(u)$ depends sufficiently smooth on $u$ and if the operator $\mathbb{A}(0)$ possesses maximal $L_{p}$-regularity (for a precise definition see the next section), then a local curve of positive solutions to (1.7), (1.8) bifurcates from the trivial branch $(\lambda, u)=(\lambda, 0), \lambda \in \mathbb{R}$. This local branch was subsequently extended in [19] to a global continuum by applying Rabinowitz' global alternative [13], but for less general diffusion operators. More precisely, the existence of an unbounded continuum of nontrivial positive solutions $(\lambda, u)$ to (1.7), (1.8) was derived under the assumption that the operator $\mathbb{A}(u)$ admits a suitable decomposition $\mathbb{A}(u)=\mathbb{A}_{0}+\mathbb{A}_{*}(u)$ with $\mathbb{A}_{*}$ being of "lower order". Although the operator $\mathbb{A}(u)$ may still depend nonlinearly on $u$, a quasilinear dependence, however, is not covered by the bifurcation result of [19]. In particular, the result of [19] does not apply to the operator from example (1.2).
The aim of this paper is to remedy this deficiency by establishing a global continuum of positive solutions to (1.7)-(1.8) for truly quasilinear diffusion operators $\mathbb{A}(u)$. After recalling (and refining) in Section 2 the results of [18] on local bifurcation, we shall show in Section 3 global bifurcation from the trivial branch provided some convexity condition (see (3.2) for details) holds implying maximal $L_{p}$-regularity of $\mathbb{A}(u)$ for each $u$. The proof relies on a recent result of Shi \& Wang [16] which is based on the results of Pejsachowicz \& Rabier [11] and is in the spirit of the unilateral global bifurcation techniques of Rabinowitz [13] or rather their interpretation by López-Gómez [10]. As we shall see then in Section 4, the convexity condition (3.2) is not necessary provided $\mathbb{A}$ and $\ell$ in (1.7), (1.8) depend real analytically on $u$. Indeed, in this case, the analytic bifurcation theory due to Buffoni \& Toland [5] yields a global smooth curve of positive solutions to (1.7), (1.8). Finally, in Section 5 we revisit problem (1.1)-(1.4) and demonstrate how the results of Section 4 may be applied in this concrete situation. Further examples to which the results of the present paper apply can be found in [18, 19, 21].

## 2. Preliminaries

2.1. General assumptions. If $E$ and $F$ are Banach spaces we write $\mathcal{L}(E, F)$ for the set of linear bounded operators from $E$ to $F$, and we put $\mathcal{L}(E):=\mathcal{L}(E, E)$. The subset thereof consisting of compact operators is denoted by $\mathcal{K}(E, F)$ and $\mathcal{K}(E)$, respectively. Isom $(E, F)$ stands for the set of topological isomorphisms $E \rightarrow F$. By $E \hookrightarrow F$ we mean that $E$ is compactly embedded in $F$.

Throughout the paper we assume that $E_{0}$ is a real Banach space ordered by a closed convex cone $E_{0}^{+}$, and $E_{1}$ is an embedded Banach space such that

$$
\begin{equation*}
\text { the embedding } E_{1} \hookrightarrow E_{0} \text { is dense and compact . } \tag{2.1}
\end{equation*}
$$

We fix $p \in(1, \infty)$ and set $E_{\zeta}:=\left(E_{0}, E_{1}\right)_{\zeta, p}$ with $(\cdot, \cdot)_{\zeta, p}$ being the real interpolation functor for $\varsigma:=\varsigma(p):=1-1 / p$. For each $\theta \in(0,1) \backslash\{1-1 / p\}$ we let $(\cdot, \cdot)_{\theta}$ denote an admissible interpolation functor, that is, an interpolation functor $(\cdot, \cdot)_{\theta}$ such that the embedding

$$
E_{1} \hookrightarrow E_{\theta}:=\left(E_{0}, E_{1}\right)_{\theta}
$$

is dense. Note that the embedding $E_{\theta} \hookrightarrow E_{\vartheta}$ is compact for $0 \leq \vartheta<\theta \leq 1$ (see [3, I.Thm.2.11.1]). The interpolation spaces $E_{\theta}, 0 \leq \theta \leq 1$ are given their natural order induced by the cone $E_{\theta}^{+}:=E_{\theta} \cap E_{0}^{+}$. We suppose that

$$
\begin{equation*}
\operatorname{int}\left(E_{\zeta}^{+}\right) \neq \varnothing \tag{2.2}
\end{equation*}
$$

i.e. $E_{\zeta}^{+}$has a non-empty interior. Recall that $a_{m} \in(0, \infty]$ and set $J:=\left[0, a_{m}\right)$. Observe that $a_{m}=\infty$ is explicitly allowed. We introduce the spaces

$$
\mathbb{E}_{0}:=L_{p}\left(J, E_{0}\right), \quad \mathbb{E}_{1}:=L_{p}\left(J, E_{1}\right) \cap W_{p}^{1}\left(J, E_{0}\right)
$$

and recall the embedding

$$
\begin{equation*}
\mathbb{E}_{1} \hookrightarrow B U C\left(J, E_{\varsigma}\right), \tag{2.3}
\end{equation*}
$$

where BUC stands for the bounded and uniformly continuous functions. Thus, the trace

$$
\gamma_{0} u:=u(0), \quad u \in \mathbb{E}_{1},
$$

yields a well-defined operator $\gamma_{0} \in \mathcal{L}\left(\mathbb{E}_{1}, E_{\zeta}\right)$. We let $\mathbb{E}_{1}^{+}:=L_{p}^{+}\left(J, E_{1}\right) \cap W_{p}^{1}\left(J, E_{0}\right)$ denote the positive cone of $\mathbb{E}_{1}$ and put $\dot{\mathbb{E}}_{1}^{+}:=\mathbb{E}_{1}^{+} \backslash\{0\}$. We fix a Banach space $\mathbb{F}$ such that

$$
\begin{equation*}
\mathbb{E}_{1} \hookrightarrow \mathbb{F} \hookrightarrow \mathbb{E}_{0} \tag{2.4}
\end{equation*}
$$

and let $\Sigma$ denote an open connected zero-neighborhood in $\mathbb{F}$. Then $\Sigma_{1}:=\Sigma \cap \mathbb{E}_{1}$ is an open connected zero-neighborhood in $\mathbb{E}_{1}$. Suppose that for some $\vartheta \in(\varsigma, 1]$ we have ${ }^{1}$

$$
\begin{equation*}
\mathbb{A} \in C^{1}\left(\Sigma, \mathcal{L}\left(\mathbb{E}_{1}, \mathbb{E}_{0}\right)\right), \quad \ell \in C^{1}\left(\Sigma, \mathcal{L}\left(\mathbb{E}_{1}, E_{\vartheta}\right)\right) \tag{2.5}
\end{equation*}
$$

and that for each $u \in \Sigma_{1}$, the operator $\mathbb{A}(u)$ possesses maximal $L_{p}$-regularity, that is,

$$
\begin{equation*}
\left(\partial_{a}+\mathbb{A}(u), \gamma_{0}\right) \in \operatorname{Isom}\left(\mathbb{E}_{1}, \mathbb{E}_{0} \times E_{\zeta}\right), \quad u \in \Sigma_{1} \tag{2.6}
\end{equation*}
$$

Then

$$
\begin{equation*}
\left(u \mapsto T[u]:=\left(\partial_{a}+\mathbb{A}(u), \gamma_{0}\right)^{-1}\right) \in C\left(\Sigma_{1}, \mathcal{L}\left(\mathbb{E}_{0} \times E_{\zeta}, \mathbb{E}_{1}\right)\right) \tag{2.7}
\end{equation*}
$$

due to continuity of the inversion map $B \mapsto B^{-1}$ for linear operators. We suppose that

$$
\begin{equation*}
T[u](0, \cdot) \in \mathcal{L}_{+}\left(E_{\zeta}, \mathbb{E}_{1}\right), \quad u \in \Sigma_{1} \tag{2.8}
\end{equation*}
$$

that is, $T[u](0, \cdot)$ maps the positive cone $E_{\varsigma}^{+}$into the positive cone $\mathbb{E}_{1}^{+}$. Set

$$
Q(u):=\ell(u) T[u](0, \cdot), \quad u \in \Sigma_{1}
$$

[^1]and note that by (2.5) and $E_{\vartheta} \hookrightarrow E_{\zeta}$ we have
\[

$$
\begin{equation*}
Q \in C\left(\Sigma_{1}, \mathcal{L}\left(E_{\zeta}, E_{\vartheta}\right)\right) \cap C\left(\Sigma_{1}, \mathcal{K}\left(E_{\zeta}\right)\right) . \tag{2.9}
\end{equation*}
$$

\]

We further suppose that

$$
\begin{align*}
& \lambda_{0}^{-1}>0 \text { is a simple eigenvalue of } Q(0) \text { with eigenvector } \Phi_{0} \in \operatorname{int}\left(E_{\zeta}^{+}\right),  \tag{2.10}\\
& \text {and there is no other eigenvalue of } Q(0) \text { with an eigenvector in } E_{\zeta}^{+} .
\end{align*}
$$

For our global bifurcation results stated in Section 3 and Section 4 we shall strengthen some of the conditions later on. We point out that the assumptions above are met in many applications, in particular in the situation of (1.1)-(1.4) (we shall be more specific in Section 5). For instance, (2.6) is satisfied for quite general elliptic second order differential operators (see e.g. [9, Thm.8.2]) and (2.8) is the maximum principle for parabolic equations [8]. Regarding assumptions (2.4) and (2.10) we note:
Remarks 2.1. (a) Let $a_{m}<\infty$. If $\alpha \in[0,1)$ and $s \in[0,1-\alpha)$, then $\mathbb{E}_{1} \hookrightarrow W_{p}^{s}\left(J, E_{\alpha}\right) \hookrightarrow \mathbb{E}_{0}$.
Proof. This follows from a generalized Aubin-Dubinskii lemma [4, Thm.1.1].
(b) If $Q(0) \in \mathcal{K}\left(E_{\zeta}\right)$ is strongly positive, i.e. if $Q(0) \Phi \in \operatorname{int}\left(E_{\zeta}^{+}\right)$for each $\Phi \in E_{\zeta}^{+} \backslash\{0\}$, then (2.10) holds with $\lambda_{0}^{-1}$ equals the spectral radius of $Q(0)$.
Proof. This is a consequence of the Krein-Rutman theorem [8, Thm.12.4].
The assumptions imposed above imply that $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ solves (1.7), (1.8) if and only if

$$
\begin{equation*}
u=T[u](0, u(0)), \quad u(0)=\lambda Q(u) u(0) \tag{2.11}
\end{equation*}
$$

which follows by plugging the solution of (1.7) - given by the first identity of (2.11) - into (1.8). Equivalently, setting $S:=T[0]$ we see that $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ solves (1.7), (1.8) if and only if

$$
u=S((\mathbb{A}(0)-\mathbb{A}(u)) u, \lambda \ell(u) u)
$$

We shall use both characterizations of solutions in the sequel. Note that (2.11) implies that $u(0)$ (if nonzero) is an eigenvector of $Q(u)$ with eigenvalue $\lambda^{-1}$.
2.2. Local Bifurcation. By what we have just observed, solving (1.7), (1.8) is equivalent to finding the zeros $(\lambda, u)$ of the function $F: \mathbb{R} \times \Sigma_{1} \rightarrow \mathbb{E}_{1}$ defined as

$$
F(\lambda, u):=u-S((\mathbb{A}(0)-\mathbb{A}(u)) u, \lambda \ell(u) u), \quad(\lambda, u) \in \mathbb{R} \times \Sigma_{1} .
$$

Let

$$
\mathfrak{S}:=\left\{(\lambda, u) \in \mathbb{R} \times \Sigma_{1} ; F(\lambda, u)=0\right\} .
$$

Clearly, $(\lambda, u)=(\lambda, 0)$ for $\lambda \in \mathbb{R}$ gives a trivial branch in $\mathfrak{S}$ of solutions to (1.7), (1.8). We shall next show that a nontrivial branch of positive solutions bifurcates from this branch at the point $(\lambda, u)=\left(\lambda_{0}, 0\right)$. For this we first show that the Fréchet derivative $F_{u}(\lambda, u)$ of $F$ with respect to $u$, given by

$$
\begin{equation*}
F_{u}(\lambda, u)[\phi]=\phi-S((\mathbb{A}(0)-\mathbb{A}(u)) \phi, \lambda \ell(u) \phi)-S\left(-\mathbb{A}_{u}(u)[\phi] u, \lambda \ell_{u}(u)[\phi] u\right), \quad \phi \in \mathbb{E}_{1}, \tag{2.12}
\end{equation*}
$$

is an index zero Fredholm operator. This will follow from the following observation:
Proposition 2.2. Let $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ be fixed and set

$$
\mathcal{F} \phi:=\mathcal{F}(\lambda, u) \phi:=\phi-S((\mathbb{A}(0)-\mathbb{A}(u)) \phi, \lambda \ell(u) \phi), \quad \phi \in \mathbb{E}_{1} .
$$

Then $\mathcal{F} \in \mathcal{L}\left(\mathbb{E}_{1}\right)$ is a Fredholm operator of index zero. More precisely,

$$
\operatorname{ker}(\mathcal{F})=\{T[u](0, w) ; w \in \operatorname{ker}(1-\lambda Q(u))\}
$$

and

$$
\begin{equation*}
\operatorname{rg}(\mathcal{F})=\left\{h \in \mathbb{E}_{1} ; h(0)+\lambda \ell(u) T[u]\left(\partial_{a} h+\mathbb{A}(0) h, 0\right) \in \operatorname{rg}(1-\lambda Q(u))\right\} \tag{2.13}
\end{equation*}
$$

with

$$
\operatorname{dim}(\operatorname{ker}(\mathcal{F}))=\operatorname{codim}(\operatorname{rg}(\mathcal{F}))=\operatorname{dim}(\operatorname{ker}(1-\lambda Q(u)))<\infty
$$

Proof. The idea of the proof is the same as in [18, Lem.2.1] and it is rather the functional analytic setting that has to be modified slightly. For the reader's ease we include a complete proof here: By definition of $S=T[0]$, the equation $\mathcal{F} \phi=h$ for $\phi, h \in \mathbb{E}_{1}$ is equivalent to

$$
\begin{align*}
\partial_{a} \phi+\mathbb{A}(u) \phi & =\partial_{a} h+\mathbb{A}(0) h,  \tag{2.14}\\
\phi(0)-\lambda \ell(u) \phi & =h(0) . \tag{2.15}
\end{align*}
$$

From (2.14) it follows

$$
\begin{equation*}
\phi=T[u]\left(\partial_{a} h+\mathbb{A}(0) h, 0\right)+T[u](0, \phi(0)) \tag{2.16}
\end{equation*}
$$

and, when plugged into (2.15), we obtain

$$
\begin{equation*}
(1-\lambda Q(u)) \phi(0)=h(0)+\lambda \ell(u) T[u]\left(\partial_{a} h+\mathbb{A}(0) h, 0\right) \tag{2.17}
\end{equation*}
$$

The statement of the proposition is trivial if $\lambda=0$, so let $\lambda \neq 0$. If $1 / \lambda$ belongs to the resolvent set of $Q(u) \in \mathcal{K}\left(E_{\zeta}\right)$, then (2.16), (2.17) entail a trivial kernel $\operatorname{ker}(\mathcal{F})=\{0\}$. Moreover, in this case, for an arbitrary $h \in \mathbb{E}_{1}$, there is a unique $\phi(0) \in E_{\zeta}$ solving (2.17) as its right hand side belongs to $E_{\zeta}$. Consequently, the corresponding $\phi \in \mathbb{E}_{1}$, given by (2.16), is the unique solution to $\mathcal{F} \phi=h$. This gives the assertion in this case.
Otherwise, if $1 / \lambda$ is an eigenvalue of $Q(u) \in \mathcal{K}\left(E_{\zeta}\right)$, then (2.16), (2.17) yield the characterization of $\operatorname{ker}(\mathcal{F})$ and $\operatorname{rg}(\mathcal{F})$ as claimed. In particular, since $T[u]$ is an isomorphism, we deduce $\operatorname{dim}(\operatorname{ker}(\mathcal{F}))=\operatorname{dim}(\operatorname{ker}(1-\lambda Q(u)))$ which is a finite number because $1 / \lambda$ is an eigenvalue of the compact operator $Q(u)$. Moreover, $\operatorname{rg}(\mathcal{F})$ is closed in $\mathbb{E}_{1}$ since $M:=\operatorname{rg}(1-\lambda Q(u))$ is closed by the compactness of $\lambda Q(u)$ and due to (2.3), (2.5), and (2.7). Next, to compute $\operatorname{codim}(\operatorname{rg}(\mathcal{F}))$ note that

$$
\operatorname{codim}(M)=\operatorname{dim}(\operatorname{ker}(1-\lambda Q(u)))<\infty
$$

hence $M$ is complemented in $E_{\zeta}$ which yields a direct sum decomposition $E_{\zeta}=M \oplus N$. Denoting by $P_{M} \in \mathcal{L}\left(E_{\zeta}\right)$ a projection onto $M$ along $N$, we set

$$
\begin{equation*}
\mathbb{P} h:=S\left(\partial_{a} h+\mathbb{A}(0) h, P_{M} h(0)-\left(1-P_{M}\right) \lambda \ell(u) T[u]\left(\partial_{a} h+\mathbb{A}(0) h, 0\right)\right), \quad h \in \mathbb{E}_{1} \tag{2.18}
\end{equation*}
$$

and obtain $\mathbb{P} \in \mathcal{L}\left(\mathbb{E}_{1}\right)$ from (2.3), (2.5), and (2.7). Since

$$
\left(\partial_{a}+\mathbb{A}(0)\right)(\mathbb{P} h)=\partial_{a} h+\mathbb{A}(0) h, \quad \gamma_{0}(\mathbb{P} h)=P_{M} h(0)-\left(1-P_{M}\right) \lambda \ell(u) T[u]\left(\partial_{a} h+\mathbb{A}(0) h, 0\right)
$$

the characterization (2.13) actually implies that $\mathbb{P}$ maps $\mathbb{E}_{1}$ into $\operatorname{rg}(\mathcal{F})$. Furthermore, if $h \in \operatorname{rg}(\mathcal{F})$, then (2.13) also ensures

$$
\mathbb{P} h=S\left(\partial_{a} h+\mathbb{A}(0) h, h(0)\right)=h
$$

so $\mathbb{P}(\operatorname{rg}(\mathcal{F}))=\operatorname{rg}(\mathcal{F})$. Thus $\mathbb{P}^{2}=\mathbb{P}$ with $\operatorname{rg}(\mathbb{P})=\operatorname{rg}(\mathcal{F})$ is a projection and

$$
\begin{equation*}
\mathbb{E}_{1}=\operatorname{ker}(\mathbb{P}) \oplus \operatorname{rg}(\mathcal{F}) \tag{2.19}
\end{equation*}
$$

Since $S$ is an isomorphism, we obtain

$$
\begin{equation*}
\operatorname{ker}(\mathbb{P})=\left\{h \in \mathbb{E}_{1} ; \partial_{a} h+\mathbb{A}(0) h=0, h(0) \in N\right\} \tag{2.20}
\end{equation*}
$$

from which we deduce $\operatorname{dim}(\operatorname{ker}(\mathbb{P}))=\operatorname{dim}(N)$ and the statement follows.
For future purposes let us explicitly state the following decomposition of $\mathbb{E}_{1}$.
Remark 2.3. The direct sum decomposition

$$
\mathbb{E}_{1}=\operatorname{span}\left(S\left(0, \Phi_{0}\right)\right) \oplus \operatorname{rg}\left(F_{u}\left(\lambda_{0}, 0\right)\right)
$$

holds.

Proof. Taking $(\lambda, u)=\left(\lambda_{0}, 0\right)$ in Proposition 2.2 and noticing that $\mathcal{F}\left(\lambda_{0}, 0\right)=F_{u}\left(\lambda_{0}, 0\right)$, the assertion follows from (2.19) and (2.20) with $N=\operatorname{ker}\left(1-\lambda_{0} Q(0)\right)=\mathbb{R} \cdot \Phi_{0}$.

Corollary 2.4. For each $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$, the Fréchet derivative $F_{u}(\lambda, u) \in \mathcal{L}\left(\mathbb{E}_{1}\right)$ is a Fredholm operator of index zero.

Proof. Set

$$
K(u) \phi:=S\left(-\mathbb{A}_{u}(u)[\phi] u, \lambda \ell_{u}(u)[\phi] u\right), \quad \phi \in \mathbb{E}_{1} .
$$

Then, by (2.4) and (2.5), $K(u)$ coincides with the Fréchet derivative

$$
K(u)=\left.D_{w} S(-\mathbb{A}(w) u, \lambda \ell(w) u)\right|_{w=u} \in \mathcal{L}\left(\mathbb{F}, \mathbb{E}_{1}\right) \subset \mathcal{K}\left(\mathbb{E}_{1}\right)
$$

Consequently, Proposition 2.2 and (2.12) show that $F_{u}(\lambda, u)$ is a compact perturbation of a Fredholm operator of index zero, so $F_{u}(\lambda, u)$ itself is a Fredholm operator of index zero.
Next, we verify that we may apply the Crandall-Rabinowitz theorem on local bifurcation for the map $F$.
Corollary 2.5. The kernel of $F_{u}\left(\lambda_{0}, 0\right)$ is one-dimensional, i.e. $\operatorname{ker}\left(F_{u}\left(\lambda_{0}, 0\right)\right)=\mathbb{R} \cdot S\left(0, \Phi_{0}\right)$, and the transversality condition $F_{\lambda, u}\left(\lambda_{0}, 0\right)\left[1, S\left(0, \Phi_{0}\right)\right] \notin \operatorname{rg}\left(F_{u}\left(\lambda_{0}, 0\right)\right)$ is satisfied.

Proof. It readily follows from (2.10), (2.12), and Proposition 2.2 that $\operatorname{ker}\left(F_{u}\left(\lambda_{0}, 0\right)\right)=\mathbb{R} \cdot S\left(0, \Phi_{0}\right)$. Moreover, (2.10) and (2.12) imply $F_{\lambda, u}\left(\lambda_{0}, 0\right)\left[1, S\left(0, \Phi_{0}\right)\right]=-\left(0, \lambda_{0}^{-1} \Phi_{0}\right)$. Suppose that

$$
-\left(0, \lambda_{0}^{-1} \Phi_{0}\right) \in \operatorname{rg}\left(F_{u}\left(\lambda_{0}, 0\right)\right)
$$

Then $\lambda_{0}^{-1} \Phi_{0} \in \operatorname{rg}\left(1-\lambda_{0} Q(0)\right)$ due to Proposition 2.2 contradicting the fact that

$$
\operatorname{rg}\left(1-\lambda_{0} Q(0)\right) \cap \operatorname{ker}\left(1-\lambda_{0} Q(0)\right)=\{0\}
$$

since $\lambda_{0}^{-1}$ is a simple eigenvalue of $Q(0)$ according to (2.10).
Based on the foregoing observations we are in a position to apply the celebrated CrandallRabinowitz theorem [6] on local bifurcation and obtain a branch in $\mathfrak{S}$ of positive solutions to (1.7), (1.8). The following result has been observed in [18].

Theorem 2.6. Assume (2.1), (2.2), (2.4), (2.6), (2.8), and (2.10). Then there are $\varepsilon>0$ and a continuous function $(\bar{\lambda}, \bar{u}):(-\varepsilon, \varepsilon) \rightarrow \mathbb{R} \times \Sigma_{1}$ such that the curves

$$
\mathfrak{K}^{ \pm}:=\{(\bar{\lambda}(t), \bar{u}(t)) ; 0 \leq \pm t<\varepsilon\} \subset \mathfrak{S}
$$

bifurcate from the trivial branch $\{(\lambda, 0) ; \lambda \in \mathbb{R}\}$ at $(\bar{\lambda}(0), \bar{u}(0))=\left(\lambda_{0}, 0\right)$ and

$$
\begin{equation*}
\bar{u}(t)=t S\left(0, \Phi_{0}\right)+o(t) \quad \text { as } t \rightarrow 0 \tag{2.21}
\end{equation*}
$$

Near the bifurcation point $\left(\lambda_{0}, 0\right)$, all nontrivial zeros of $F$ lie on the curve $\mathfrak{K}^{-} \cup \mathfrak{K}^{+}$. Moreover,

$$
\mathfrak{K}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\} \subset(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}
$$

and $\mathfrak{K}^{-} \cap(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}=\varnothing$.
Proof. According to Corollary 2.4, Corollary 2.5, and [6] we only have to prove the statements on positivity. From (2.21) and (2.10) it follows that

$$
t^{-1} \gamma_{0} \bar{u}(t)=\Phi_{0}+\gamma_{0} \frac{o(t)}{t} \in \operatorname{int}\left(E_{\zeta}^{+}\right) \quad \text { as } t \rightarrow 0
$$

whence $\gamma_{0} \bar{u}(t) \in E_{\varsigma}^{+}$provided $t \in(0, \varepsilon)$ is sufficiently small. Since

$$
\bar{u}(t)=T[\bar{u}(t)]\left(0, \gamma_{0} \bar{u}(t)\right),
$$

we conclude $\bar{u}(t) \in \dot{\mathbb{E}}_{1}^{+}$from (2.8) and $\bar{\lambda}(t)>0$ from the assumption $\lambda_{0}>0$ for $t \in(0, \varepsilon)$. Consequently, $\mathfrak{K}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\} \subset(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$. The same argument shows that $\mathfrak{K}^{-} \cap(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$ is empty.

Note that if $\mathbb{A}$ and $\ell$ in (2.6) are real analytic, then so is the curve $(\bar{\lambda}, \bar{u}):(-\varepsilon, \varepsilon) \rightarrow \mathbb{R} \times \Sigma_{1}$ since $F: \mathbb{R} \times \Sigma_{1} \rightarrow \mathbb{E}_{1}$ is real analytic in this case.
The direction of bifurcation may be determined from the second relation (2.11). For instance, if $Q(u) \in \mathcal{K}\left(E_{\zeta}\right)$ is strongly positive for $u \in \Sigma_{1}$, then (2.11) implies $\lambda r(Q(u))=1$ for any positive solution $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ of (1.7), (1.8) according to the Krein-Rutman theorem. Conditions may then be imposed which ensure $r(Q(u)) \leq 1$ so that bifurcation is necessarily supercritical. For further details we refer to [18].

We next prove a compactness property of the solution set $\mathfrak{S}$ that we shall use in the coming subsections.

Lemma 2.7. Any bounded and closed subset of $\mathfrak{S}$ is compact in $\mathbb{R} \times \mathbb{E}_{1}$.
Proof. Let $\left(\lambda_{n}, u_{n}\right)_{n \in \mathbb{N}}$ be any sequence in a closed subset of $\mathfrak{S}$ such that $\left\|\left(\lambda_{n}, u_{n}\right)\right\|_{\mathbb{R} \times \mathbb{E}_{1}} \leq c_{0}$ for all $n \in \mathbb{N}$ and some $c_{0} \in \mathbb{R}$. Then

$$
\begin{equation*}
u_{n}=T\left[u_{n}\right]\left(0, u_{n}(0)\right), \quad u_{n}(0)=\lambda_{n} Q\left(u_{n}\right) u_{n}(0) \tag{2.22}
\end{equation*}
$$

Due to (2.4), we may assume without loss of generality that $\left(\lambda_{n}, u_{n}\right) \rightarrow(\lambda, u)$ in $\mathbb{R} \times \Sigma$. According to (2.5) and (2.9) this implies $T\left[u_{n}\right](0, \cdot) \rightarrow T[u](0, \cdot)$ in $\mathcal{L}\left(E_{G}, \mathbb{E}_{1}\right)$ and $Q\left(u_{n}\right) \rightarrow Q(u)$ in $\mathcal{L}\left(E_{\zeta}, E_{\vartheta}\right)$. Also note that (2.3) entails $\left\|u_{n}(0)\right\|_{E_{\zeta}} \leq c$ for $n \in \mathbb{N}$. Consequently, from (2.22) we derive

$$
\begin{equation*}
\left\|u_{n}(0)\right\|_{E_{\vartheta}} \leq\left|\lambda_{n}\right|\left\|Q\left(u_{n}\right)\right\|_{\mathcal{L}\left(E_{\zeta}, E_{\vartheta}\right)}\left\|u_{n}(0)\right\|_{E_{\zeta}} \leq c, \quad n \in \mathbb{N}, \tag{2.23}
\end{equation*}
$$

and we thus may assume without loss of generality that $u_{n}(0) \rightarrow v$ in $E_{\zeta}$ since $E_{\vartheta} c E_{\zeta}$. Now (2.22) shows that $v=\lambda Q(u) v$ and $u_{n}=T\left[u_{n}\right]\left(0, u_{n}(0)\right) \rightarrow T[u](0, v)$ in $\mathbb{E}_{1}$. Clearly, $u=T[u](0, v)$ since $u_{n} \rightarrow u$ in $\mathbb{F}$ and we conclude that $\left(\lambda_{n}, u_{n}\right) \rightarrow(\lambda, u)$ in $\mathbb{R} \times \mathbb{E}_{1}$. This proves the assertion.

## 3. Global Continua

We shall show that the local curve $\mathfrak{K}^{+}$provided by Theorem 2.6 is contained in a global continuum of positive solutions to (1.7), (1.8). In this section we make use of the unilateral global bifurcation theory in the spirit of Rabinowitz's alternative [13, 10] as proposed in [16]. In Section 4 we shall give a slightly different approach by means of analytic bifurcation theory [5]. In order to apply the results of [16] we have to strengthen certain conditions. More precisely, in the following we suppose in addition to the assumptions stated in Section 2 that

$$
\begin{equation*}
E_{0}^{\prime} \text { and } E_{1} \text { are separable } \tag{3.1}
\end{equation*}
$$

and we strengthen assumption (2.6) to a convexity condition

$$
\begin{equation*}
\left(\partial_{a}+[(1-\alpha) \mathbb{A}(0)+\alpha \mathbb{A}(u)], \gamma_{0}\right) \in \operatorname{Isom}\left(\mathbb{E}_{1}, \mathbb{E}_{0} \times E_{\zeta}\right), \quad u \in \Sigma_{1}, \quad \alpha \in[0,1] \tag{3.2}
\end{equation*}
$$

We also assume that

$$
\begin{equation*}
\ell(0) \in \mathcal{L}_{+}\left(\mathbb{E}_{1}, E_{\zeta}\right) \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
Q(0) \in \mathcal{K}\left(E_{\zeta}\right) \text { is strongly positive . } \tag{3.4}
\end{equation*}
$$

According to Remark 2.1 this last assumption implies (2.10) with $\lambda_{0}^{-1}$ given by the spectral radius of $Q(0)$. Note that $E_{0}$ is separable since $E_{1}$ is separable and dense in $E_{0}$. Also note that if
$E_{0}$ is reflexive (and separable), then $E_{0}^{\prime}$ is separable.
Assumption (3.1) implies the following lemma, which is needed in order to apply the result of $[16, \S 4]$.

Lemma 3.1. $\mathbb{E}_{1}$ can be equipped with an equivalent norm which is differentiable at any point different from 0 .

Proof. Due to [15], as $\mathbb{E}_{1}$ is separable since $E_{1}$ is, the statement that $\mathbb{E}_{1}$ can be equipped with an equivalent norm which is differentiable at any point different from 0 is equivalent to saying that the dual space $\mathbb{E}_{1}^{\prime}$ is separable. But since $E_{0}^{\prime}$ is separable, the space $\mathbb{E}_{0}^{\prime}=L_{p^{\prime}}\left(J, E_{0}^{\prime}\right)$ (with $\left.1=1 / p+1 / p^{\prime}\right)$ is also separable and, moreover, densely injected in $\mathbb{E}_{1}^{\prime}$ since $\mathbb{E}_{1}$ is densely injected in $\mathbb{E}_{0}$. Whence $\mathbb{E}_{1}^{\prime}$ is separable.

The convexity condition (3.2) yields:
Lemma 3.2. For $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ and $\alpha \in[0,1]$, the operator

$$
(1-\alpha) F_{u}(\lambda, 0)+\alpha F_{u}(\lambda, u) \in \mathcal{L}\left(\mathbb{E}_{1}\right)
$$

is Fredholm of index zero.
Proof. Let $(\lambda, u) \in \mathbb{R} \times \Sigma_{1}$ and $\alpha \in[0,1]$. Noticing that

$$
\begin{aligned}
(1-\alpha) F_{u}(\lambda, 0)[\phi]+\alpha F_{u}(\lambda, u)[\phi]= & \phi-S(\alpha(\mathbb{A}(0)-\mathbb{A}(u)) \phi, \lambda[(1-\alpha) \ell(0)+\alpha \ell(u)] \phi) \\
& -\alpha S\left(-\mathbb{A}_{u}(u)[\phi] u, \lambda \ell_{u}(u)[\phi] u\right)
\end{aligned}
$$

for $\phi \in \mathbb{E}_{1}$, the proof is the same as in Proposition 2.2 and Corollary 2.4 by taking (3.2) into account.

Now we can prove that there is a global continuum of positive solutions to (1.7), (1.8).
Theorem 3.3. Assume (2.1), (2.2), (2.4), (2.5), (2.8), and (3.1)-(3.4). Then there is a connected component $\mathfrak{C}^{+}$of $\overline{\mathfrak{S}}$ containing the branch $\mathfrak{K}^{+}$such that $\mathfrak{C}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\} \subset(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$. Moreover, one of the alternatives
(i) $\mathfrak{C}^{+}$is unbounded in $\mathbb{R} \times \mathbb{E}_{1}$, or
(ii) $\mathfrak{C}^{+}$intersects with the boundary $\mathbb{R} \times \partial \Sigma_{1}$
occurs. In particular, if $\Sigma=\mathbb{F}$, then $\mathfrak{C}^{+}$is unbounded in $\mathbb{R} \times \mathbb{E}_{1}$.
Proof. Due to Lemma 3.1 and Lemma 3.2, we may apply [16, Thm.4.4] and deduce that $\mathfrak{K}^{+}$is contained in a connected component $\mathfrak{C}^{+}$of $\overline{\mathfrak{S}}$ and one of the alternatives
(a) $\mathfrak{C}^{+}$is not compact in $\mathbb{R} \times \Sigma_{1}$, or
(b) $\mathfrak{C}^{+}$contains a point $\left(\lambda_{*}, 0\right)$ with $\lambda_{*} \neq \lambda_{0}$, or
(c) $\mathfrak{C}^{+}$contains a point $(\lambda, z)$ with $z \neq 0$ and $z \in \operatorname{rg}\left(F_{u}\left(\lambda_{0}, 0\right)\right)$
occurs, where we have used in (c) the decomposition $\mathbb{E}_{1}=\operatorname{span}\left(S\left(0, \Phi_{0}\right)\right) \oplus \operatorname{rg}\left(F_{u}\left(\lambda_{0}, 0\right)\right)$ stated in Remark 2.3. Notice that, owing to Lemma 2.7 (see [16, Rem.4.2]), alternative (a) is equivalent to saying that
(i) $\mathfrak{C}^{+}$is unbounded in $\mathbb{R} \times \mathbb{E}_{1}$, or
(ii) $\mathfrak{C}^{+}$intersects with the boundary $\mathbb{R} \times \partial \Sigma_{1}$.

According to Theorem 2.6 , the component $\mathfrak{C}^{+}$near the bifurcation point $\left(\lambda_{0}, 0\right)$ coincides with $\mathfrak{K}^{+}$. Next, we show that $\mathfrak{C}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\} \subset(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$. Indeed, if $\mathfrak{C}^{+}$leaves $(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$at some point $(\lambda, u) \in \mathfrak{C}^{+} \cap \mathbb{R} \times \mathbb{E}_{1}$ with $(\lambda, u) \notin(0, \infty) \times \dot{E}_{1}^{+}$, there is a sequence $\left(\left(\lambda_{j}, u_{j}\right)\right)_{j \in \mathbb{N}}$ in $\mathfrak{C}^{+} \cap(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$such that $\left(\lambda_{j}, u_{j}\right) \rightarrow(\lambda, u)$ in $\mathbb{R} \times \mathbb{E}_{1}$. Clearly, $\lambda \geq 0$ and $u \in \mathbb{E}_{1}^{+}$with $\lambda=0$
or $u \equiv 0$. But since $(\lambda, u) \in \mathfrak{S}$, we readily deduce from (2.11) that $\lambda=0$ implies $u \equiv 0$. Hence $u \equiv 0$ in any case, i.e. $\left(\lambda_{j}, u_{j}\right) \rightarrow(\lambda, 0)$ in $\mathbb{R} \times \mathbb{E}_{1}$. Again by (2.11), we have

$$
\begin{equation*}
u_{j}=T\left[u_{j}\right]\left(0, u_{j}(0)\right), \quad u_{j}(0)=\lambda_{j} Q\left(u_{j}\right) u_{j}(0) \tag{3.5}
\end{equation*}
$$

Since $v_{j}:=u_{j} /\left\|u_{j}\right\|_{\mathbb{E}_{1}}$ defines a bounded sequence in $\mathbb{E}_{1}$, by (2.4) we may extract a subsequence of $\left(v_{j}\right)$ (which we do not index) which converges to some $v$ in $\mathbb{F}$. From (2.7) and (2.9) we deduce $T\left[u_{j}\right] \rightarrow T[0]=S$ in $\mathcal{L}\left(\mathbb{E}_{0} \times E_{\zeta}, \mathbb{E}_{1}\right)$ and $Q\left(u_{j}\right) \rightarrow Q(0)$ in $\mathcal{L}\left(E_{\zeta}, E_{\vartheta}\right)$. As in (2.23) we then obtain from (2.9) and (2.3) that

$$
\left\|u_{j}(0)\right\|_{E_{\vartheta}} \leq c\left\|u_{j}(0)\right\|_{E_{\varsigma}} \leq c\left\|u_{j}\right\|_{\mathbb{E}_{1}}
$$

from which we conclude that the sequence $\left(v_{j}(0)\right)$ is bounded in $E_{\vartheta} \hookrightarrow E_{\zeta}$. So, extracting a further subsequence (which we again do not index) we see that $v_{j}(0) \rightarrow w$ in $E_{\zeta}^{+}$. Letting $j \rightarrow \infty$ in (3.5) yields

$$
v=S(0, w), \quad w=\lambda Q(0) w
$$

from which we first deduce that $\lambda>0$ since otherwise $w=0$ implying the contradiction $v \equiv 0$. Consequently, $w \in E_{\zeta}^{+}$is an eigenvector of $Q(0)$ to the eigenvalue $1 / \lambda$. Thus $\lambda=\lambda_{0}$ and $w=\alpha \Phi_{0}$ for some $\alpha>0$ according to (2.10), hence $(\lambda, u)=\left(\lambda_{0}, 0\right)$. Therefore, $\mathfrak{C}^{+}$leaves the set $(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$only at $\left(\lambda_{0}, 0\right)$. Thus $\mathfrak{C}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\}$ is contained in $(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$. In particular, alternative (b) above does not occur. We finally show that alternative (c) does not occur as well. Suppose to the contrary that $\mathfrak{C}^{+}$contains a point $(\lambda, z)$ with $z \neq 0$ and $z=F_{u}\left(\lambda_{0}, 0\right) \zeta$ for some $\zeta \in \mathbb{E}_{1}$. Then $z \in \dot{\mathbb{E}}_{1}^{+}$and $\zeta-z=S\left(0, \lambda_{0} \ell(0) \zeta\right)$. Recall from Corollary 2.5 that $\phi_{*}:=S\left(0, \Phi_{0}\right)$ with $\Phi_{0}$ from (2.10) satisfies $\phi_{*}=S\left(0, \lambda_{0} \ell(0) \phi_{*}\right)$. Since $\Phi_{0} \in \operatorname{int}\left(E_{\zeta}^{+}\right)$, we find $\kappa>0$ such that $\kappa \Phi_{0}+\zeta(0)-z(0)$ belongs to $E_{\varsigma}^{+}$. Set $\psi:=\kappa \phi_{*}+\zeta-z$. Due to $\psi=\lambda_{0} S(0, \ell(0)(\psi+z))$ we conclude $\partial_{a} \psi+\mathbb{A}(0) \psi=0$ on the one hand from which $\psi=S(0, \psi(0))$, and

$$
\psi(0)=\lambda_{0} \ell(0) \psi+\lambda_{0} \ell(0) z
$$

on the other. Combining these two observations we derive the equation

$$
\begin{equation*}
\left(1-\lambda_{0} Q(0)\right) \psi(0)=\lambda_{0} \ell(0) z \tag{3.6}
\end{equation*}
$$

However, since $\lambda_{0}^{-1}$ is the spectral radius of the strongly positive compact operator $Q(0)$ and since $\lambda_{0} \ell(0) z \in E_{G}^{+}$by (3.4), equation (3.6) has no positive solution according to [8, Cor.12.4] contradicting $\psi(0)=\kappa \Phi_{0}+\zeta(0)-z(0) \in E_{\zeta}^{+}$. Therefore, alternative (c) above is impossible and the theorem is proven.

## 4. Global Branches in the Analytic Case

In this section we shall show that assumptions (3.1)-(3.4) of Theorem 3.3 are not needed to extend the function $(\bar{\lambda}, \bar{u})$ from $(0, \varepsilon)$ to $(0, \infty)$ provided that $\mathbb{A}$ and $\ell$ are real analytic. In this case we obtain a slightly better result. So, let us strengthen assumption (2.5) to

$$
\begin{equation*}
\mathbb{A} \in C^{\omega}\left(\Sigma, \mathcal{L}\left(\mathbb{E}_{1}, \mathbb{E}_{0}\right)\right), \quad \ell \in C^{\omega}\left(\Sigma, \mathcal{L}\left(\mathbb{E}_{1}, E_{\vartheta}\right)\right) . \tag{4.1}
\end{equation*}
$$

As noted in Section 2, the function $(\bar{\lambda}, \bar{u})$ from Theorem 2.6 is real analytic in this case.
Theorem 4.1. Assume (2.1), (2.2), (2.4), (2.6), (2.8), (2.10), and (4.1). Then there is a continuous curve $\mathfrak{R}^{+}=\{(\bar{\lambda}(t), \bar{u}(t)) ; t \in[0, \infty)\} \subset \mathfrak{S}$ extending $\mathfrak{K}^{+}$. The curve $\mathfrak{R}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\}$ lies in $(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$ and has at each point a local analytic and injective reparametrization. Moreover, one of the alternatives
(i) $\|(\bar{\lambda}(t), \bar{u}(t))\|_{\mathbb{R} \times \mathbb{E}_{1}} \rightarrow \infty$ as $t \rightarrow$, or
(ii) $\bar{u}(t) \rightarrow \partial \Sigma_{1}$ as $t \rightarrow \infty$
occurs. In particular, if $\Sigma=\mathbb{F}$, then (i) occurs.

Proof. Due to Corollary 2.4, Corollary 2.5, and Lemma 2.7 we may apply [5, Thm.9.1.1]. Consequently, there is a continuous curve $\mathfrak{R}^{+}=\{(\bar{\lambda}(t), \bar{u}(t)) ; t \in[0, \infty)\} \subset \mathfrak{S}$ extending $\mathfrak{K}^{+}$and having a local analytic and injective reparametrization. For this curve $\mathfrak{R}^{+}$, one of the alternatives
(i) $\|(\bar{\lambda}(t), \bar{u}(t))\|_{\mathbb{R} \times \mathbb{E}_{1}} \rightarrow \infty$ as $t \rightarrow$, or
(ii) $\bar{u}(t) \rightarrow \partial \Sigma_{1}$ as $t \rightarrow \infty$, or
(iii) $\mathfrak{R}^{+}$is a closed loop, i.e. there is a minimal $\tau>0$ such that $\mathfrak{R}^{+}=\{(\bar{\lambda}(t), \bar{u}(t)) ; 0 \leq t \leq \tau\}$ and $(\bar{\lambda}(\tau), \bar{u}(\tau))=(\bar{\lambda}(0), \bar{u}(0))=\left(\lambda_{0}, 0\right)$,
occurs. Moreover, if $\left(\bar{\lambda}\left(t_{1}\right), \bar{u}\left(t_{1}\right)\right)=\left(\bar{\lambda}\left(t_{2}\right), \bar{u}\left(t_{2}\right)\right)$ for some $t_{1} \neq t_{2}$ with

$$
\operatorname{ker}\left(F_{u}\left(\bar{\lambda}\left(t_{1}\right), \bar{u}\left(t_{1}\right)\right)\right)=\{0\}
$$

then (iii) occurs and $\left|t_{1}-t_{2}\right|$ is an integer multiple of $\tau$. Finally, the set

$$
\left\{t \geq 0 ; \operatorname{ker}\left(F_{u}(\bar{\lambda}(t), \bar{u}(t))\right) \neq\{0\}\right\}
$$

has no accumulation points. So, the assertion follows provided we can prove that the curve $\mathfrak{R}^{+} \backslash\left\{\left(\lambda_{0}, 0\right)\right\}$ lies in $(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$and that alternative (iii) does not occur. For this we use an argument similar to [5, Thm.9.2.2]: Set

$$
T_{*}:=\sup \left\{T>0 ;(\bar{\lambda}(t), \bar{u}(t)) \in(0, \infty) \times \dot{\mathbb{E}}_{1}^{+} \text {for } 0<t<T\right\}
$$

and note that $T_{*} \geq \varepsilon$ according to Theorem 2.6. Assuming $T_{*}<\infty$, there is a sequence $\left(t_{j}\right)$ with $t_{j} \nearrow T_{*}$ such that $\left(\lambda_{j}, u_{j}\right):=\left(\bar{\lambda}\left(t_{j}\right), \bar{u}\left(t_{j}\right)\right) \in(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$converges to $(\lambda, u):=\left(\bar{\lambda}\left(T_{*}\right), \bar{u}\left(T_{*}\right)\right)$ in $\mathbb{R} \times \mathbb{E}_{1}$ and $(\lambda, u) \notin(0, \infty) \times \dot{\mathbb{E}}_{1}^{+}$. But then, the same argument as in the proof of Theorem 3.3 yields $\left(\bar{\lambda}\left(T_{*}\right), \bar{u}\left(T_{*}\right)\right)=\left(\lambda_{0}, 0\right)$. Since Theorem 2.6 implies that the bifurcation curve which lies in $\mathbb{R}^{+} \times \mathbb{E}_{1}^{+}$and passes through $\left(\lambda_{0}, 0\right)$ is near this point uniquely determined by $\mathfrak{K}^{+}$, we derive that $(\bar{\lambda}(t), \bar{u}(t))$ belongs to $\mathfrak{K}^{+}$for $t$ less, but close to $T_{*}$. Consequently, there are sequences $r_{k} \searrow 0$ and $s_{k} \searrow 0$ such that $\left(\bar{\lambda}\left(r_{k}\right), \bar{u}\left(r_{k}\right)\right)=\left(\bar{\lambda}\left(T_{*}-s_{k}\right), \bar{u}\left(T_{*}-s_{k}\right)\right)$ and $\operatorname{ker}\left(F_{u}\left(\bar{\lambda}\left(r_{k}\right), \bar{u}\left(r_{k}\right)\right)\right)=\{0\}$. But then, as stated above, the minimally chosen $\tau>0$ with $\left(\lambda_{0}, 0\right)=(\bar{\lambda}(\tau), \bar{u}(\tau))$ divides $T_{*}-s_{k}-r_{k}$ for each $k \in \mathbb{N}$. This is obviously impossible. Therefore, $T_{*}=\infty$ and alternative (iii) above does not occur.

Note that it is not claimed in Theorem 4.1 that $\mathfrak{R}^{+}$is a maximal connected subset of $\mathfrak{S}$. Other curves or manifolds in $\mathfrak{S}$ may intersect $\mathfrak{R}^{+}$. We also point out that alternative (i) in Theorem 4.1 is stronger than saying that $\mathfrak{R}^{+}$is unbounded in $\mathbb{R} \times \mathbb{E}_{1}$ (see Theorem 3.3).

## 5. Example

We apply Theorem 3.3 to the example given in the introduction of this paper:

$$
\begin{array}{lll}
\partial_{a} u-\operatorname{div}_{x}\left(d(U(x), a, x) \nabla_{x} u\right)+\mu(U(x), a, x) u=0, & a \in\left(0, a_{m}\right), & x \in \Omega, \\
u(0, x)=\lambda \int_{0}^{a_{m}} b(U(x), a, x) u(a, x) \mathrm{d} a, & & x \in \Omega, \\
\partial_{v} u(a, x)=0, & a \in\left(0, a_{m}\right), & x \in \partial \Omega, \\
U(x)=\int_{0}^{a_{m}} u(a, x) \mathrm{d} a, & x \in \partial \Omega, \tag{5.4}
\end{array}
$$

where $\Omega \subset \mathbb{R}^{n}$ is a bounded and smooth domain and $a_{m} \in(0, \infty)$. Let $J=\left[0, a_{m}\right)$. Fix $p \in(n+2, \infty)$ and set

$$
E_{1}:=W_{p, N}^{2}(\Omega):=\left\{v \in W_{p}^{2}(\Omega) ; \partial_{v} v=0\right\} \hookrightarrow E_{0}:=L_{p}(\Omega)
$$

and

$$
\mathbb{E}_{1}:=L_{p}\left(J, W_{p, N}^{2}(\Omega)\right) \cap W_{p}^{1}\left(J, L_{p}(\Omega)\right), \quad \mathbb{E}_{0}:=L_{p}\left(J, L_{p}(\Omega)\right)
$$

Then (2.1) and (3.1) hold. Observe that the interpolation result [17, Thm. 4.3.3] and Sobolev's embedding theorem imply

$$
\begin{equation*}
E_{1}=W_{p, N}^{2}(\Omega) \hookrightarrow E_{\zeta}:=\left(L_{p}(\Omega), W_{p, N}^{2}\right)_{1-1 / p, p} \doteq W_{p, N}^{2(1-1 / p)} \hookrightarrow C^{1}(\bar{\Omega}) \tag{5.5}
\end{equation*}
$$

Thus $\operatorname{int}\left(E_{\varsigma}^{+}\right) \neq \varnothing$ while Remark 2.1 implies that

$$
\mathbb{E}_{1} \hookrightarrow \mathbb{F}:=W_{p}^{s}\left(J, E_{\vartheta}\right)
$$

for some $\vartheta>1-1 / p=\varsigma$ and some $s \in(0,1 / p)$, where $E_{\vartheta} \doteq W_{p, N}^{2 \vartheta}(\Omega)$. Let the functions $d: \mathbb{R} \times\left[0, a_{m}\right] \times \bar{\Omega} \rightarrow(\underline{d}, \infty)$ and $\mu, b: \mathbb{R} \times\left[0, a_{m}\right] \times \bar{\Omega} \rightarrow \mathbb{R}^{+}$be smooth with $\underline{d}>0$ and $b(0, \cdot, \cdot)>0$ on $\left(0, a_{m}\right) \times \bar{\Omega}$. For $u \in \mathbb{F} \hookrightarrow L_{1}\left(J, E_{\vartheta}\right)$, set

$$
U:=\int_{0}^{a_{m}} u(a, \cdot) \mathrm{d} a \in E_{\vartheta}
$$

and define

$$
\mathbb{A}(u, a) w:=-\operatorname{div}_{x}\left(d(U, a, x) \nabla_{x} w\right)+\mu(U, a, x) w, \quad w \in E_{1}, \quad u \in \mathbb{F}, \quad a \in J, \quad x \in \Omega .
$$

Then, by [22, Prop.4.1],

$$
\mathbb{A} \in C^{1}\left(\mathbb{F}, L_{\infty}\left(J, \mathcal{L}\left(W_{p, N}^{2}(\Omega), L_{p}(\Omega)\right)\right)\right)
$$

and thus in particular $\mathbb{A} \in C^{1}\left(\mathbb{F}, \mathcal{L}\left(\mathbb{E}_{1}, \mathbb{E}_{0}\right)\right)$. Setting

$$
\lambda \ell(v) u:=\int_{0}^{a_{m}} \lambda b(v, a, \cdot) u(a) \mathrm{d} a, \quad v \in \mathbb{F}, \quad u \in \mathbb{E}_{1},
$$

we have $\ell \in C^{1}\left(\mathbb{F}, \mathcal{L}\left(\mathbb{E}_{1}, E_{\vartheta}\right)\right)$ by [22, Prop.4.1] together with the multiplication result of [2, Thm.4.1]. Thus condition (2.5) holds. Note that for $\alpha \in[0,1], u \in \mathbb{F}$ and $w \in E_{1}$ we have

$$
\begin{aligned}
\mathbb{A}_{\alpha}(u, \cdot) w: & =(1-\alpha) \mathbb{A}(0, \cdot) w+\alpha \mathbb{A}(u, \cdot) w \\
& =-\operatorname{div}_{x}\left([(1-\alpha) d(0, \cdot \cdot \cdot)+\alpha d(U, \cdot \cdot \cdot)] \nabla_{x} w\right)+[(1-\alpha) \mu(0, \cdot, \cdot)+\alpha \mu(U, \cdot \cdot \cdot)] w
\end{aligned}
$$

with

$$
(1-\alpha) d(0, \cdot, \cdot)+\alpha d(U, \cdot, \cdot) \geq \underline{d} .
$$

Hence, for $\alpha \in[0,1], u \in \mathbb{F}$, and $a \in J$ the operator $-\mathbb{A}_{\alpha}(u, a)$ is resolvent positive, generates a contraction semigroup on each $L_{q}(\Omega), 1<q<\infty$ (see [1]), and is self-adjoint in $L_{2}(\Omega)$. Hence [3, III.Ex.4.7.3,III.Thm.4.10.10] entail (3.2). Since for $u \in \mathbb{F}$ fixed, the mapping

$$
\mathbb{A}(u, \cdot):\left[0, a_{m}\right] \rightarrow \mathcal{L}\left(W_{p, N}^{2}(\Omega), L_{p}(\Omega)\right)
$$

is Hölder continuous, there is a unique positive evolution operator $\Pi_{u}(a, \sigma), 0 \leq \sigma \leq a \leq a_{m}$ on $E_{0}$ corresponding to $\mathbb{A}(u, \cdot)$, see [3, II.Cor.4.4.2.,II.Thm.6.4.2]. In particular, $T[u](0, \cdot)=\Pi_{u}(\cdot, 0) \in$ $\mathcal{L}_{+}\left(E_{\zeta}, \mathbb{E}_{1}\right)$ for $u \in \mathbb{F}$ and $\ell(0) \in \mathcal{L}_{+}\left(\mathbb{E}_{1}, E_{\zeta}\right)$, that is, (2.8) and (3.3) hold. Also note that the maximum principle ensures that $\Pi_{0}(a, 0) \in \mathcal{K}\left(E_{\zeta}\right)$ is strongly positive for each $a \in J \backslash\{0\}$, see [8, Sect.13]. Since $b(0, \cdot, \cdot)>0$ we conclude (see [18, Sect.3]) that

$$
Q(0)=\int_{0}^{a_{m}} b(0, a, \cdot) \Pi_{0}(a, 0) \mathrm{d} a \in \mathcal{K}\left(E_{\zeta}\right)
$$

is strongly positive, whence (3.4). Consequently, we are in a position to apply Theorem 3.3 and deduce that there is an unbounded continuum of positive solutions $(\lambda, u)$ in $(0, \infty) \times \mathbb{E}_{1}^{+}$ to (5.1)-(5.4).

Let us point out that this is just one example and can be extended in various ways. For instance, one may consider more general (uniformly elliptic) differential operators subject to other boundary conditions, see [9, Thm.8.2]. Also the regularity assumptions are not chosen optimally and the phase space $E_{0}$ can be any $L_{q}(\Omega)$ provided $q \in(1, \infty)$. In Theorem 4.1 it is possible, in
principle, to take $E_{0}=L_{1}(\Omega)$, where one may check the analyticity condition (4.1) with the help of $[14,5 . T h m .4]$. It is also worthwhile to point out that, instead of using the concept of maximal $L_{p}$-regularity (see, in particular, assumption (2.6)), one may use other concepts like maximal Hölder regularity [3]. We refrain from giving details and refer to [18, 19] for other examples. For more concrete applications of global bifurcation results we refer e.g. to [21].

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[^1]:    ${ }^{1}$ Observe that this notation includes that $A=A(u, a)$ in (1.5) depends in a local way on age $a$.

