ON THE SOLVABILITY OF A MATHEMATICAL MODEL FOR PRION PROLIFERATION

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ABSTRACT. We show that a model describing the interaction between normal and infectious prion proteins admits global solutions. More precisely, supposing the involved degradation rates to be bounded, we prove global existence and uniqueness of classical solutions. Based on this existence theory, we provide sufficient conditions for the existence of global weak solutions in the case of unbounded splitting rates. Moreover, we prove global stability of the disease-free steady state.

1. Introduction

The present paper aims at investigating mathematically a recent model that describes the dynamics of prion proliferation. Prions seem to be widely regarded as the infectious agent causing fatal diseases known as bovine spongiform encephalopathy (BSE) for cattle, scrapie for sheep, or Kuru and Creutzfeld-Jacob for humans. In this theory, prions are thought to be a polymeric form of a normal protein monomer PrP^C. The polymeric infectious prions PrP^{Sc} have a tendency to attach units of PrP^C in a stringlike formation, converting the latter to the infectious form. This mechanism makes PrP^{Sc} polymers more stable and is called nucleated polymerization. Above some critical size, PrP^{Sc} is very stable and polymerizes rapidly to form chains, possibly involving several thousands of monomer units. Nevertheless, PrP^{Sc} prions also can split, usually into smaller infectious prions. However, if a polymer falls below the critical size, it degrades immediately into PrP^C monomers. A model for nucleated polymerization has recently been proposed in [4], [5] (see also the references therein) describing the mechanism by which prions are hypothesized to replicate. Denoting the number of PrP^C monomers at time $t \geq 0$ by $v(t) \geq 0$ and introducing a population density $u = u(t,y) \geq 0$ for the infectious PrP^{Sc} polymers at time $t \geq 0$ and size y greater than the minimum length $y_0 > 0$, the interaction of the PrP^C monomers and the PrP^{Sc} polymers can be described by the coupled system consisting of the ordinary differential equation

$$\dot{v} = \lambda - \gamma\, v - \tau\, v \int_{y_0}^{\infty} u(t,y)\,\mathrm{d}y + 2\int_{y_0}^{\infty} u(t,y)\,\beta(y) \int_0^{y_0} y'\,\kappa(y',y)\,\mathrm{d}y'\mathrm{d}y \tag{1}$$

and the partial differential equation

$$\partial_t u + \tau\, v(t)\,\partial_y u = -\big(\mu(y) + \beta(y)\big)\,u(y) + 2\int_y^{\infty} \beta(y')\,\kappa(y,y')\,u(y')\,\mathrm{d}y' \tag{2}$$

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for $y \in (y_0, \infty)$ subject to the boundary condition

$$u(t, y_0) \,=\, 0 \;, \quad t > 0 \;. \tag{3}$$

These equations are supplemented with the initial conditions

$$v(0) \,=\, v^0 \;, \quad u(0, y) \,=\, u^0(y) \;, \quad y \in (y_0, \infty) \;. \tag{4}$$

Equation (1) includes a source term $\lambda \geq 0$, while the term $-\gamma v(t)$, with $\gamma \geq 0$, takes into account metabolic degradation of monomers. The constant $\tau > 0$ denotes the polymerization rate. Moreover, $\beta(y) \geq 0$ is the length-dependent fragmentation rate of polymers of size $y > y_0$, and $\kappa(y', y)$ is the probability of a polymer of size $y > y_0$ splitting into two pieces $y' < y$ and $y - y' < y$. The transport term $\tau v(t) \partial_y u(t, y)$ in equation (2) accounts for the loss of polymers of size y due to lengthening. A loss of polymers according to metabolic degradation is reflected by the term $-\mu(y) u(y)$. Finally, the terms involving β on the right hand side of equation (2) represent the loss and gain of PrP^{Sc} polymers caused by splitting. For a more detailed explanation of each process we refer to [4], [5] and the references therein.

Let us point out that (1), (2) is a coupled system of non-linear, non-local equations. In order to solve this equations we employ Kato's theory for hyperbolic evolution equations. That is, given a function v with appropriate regularity properties, we construct an evolution system for the partial differential equation (2). We should remark that in the absence of the kernel operator on the right hand side of (2), an evolution system can readily be obtained by using the method of characteristics.
It should also be pointed out that equations (1), (2) can be handled as an abstract quasilinear hyperbolic system in order to obtain local existence, see for instance [9, §6.4]. However, this approach does not seem to yield optimal results for equations (1), (2).

Before outlining the contents of this paper, we summarize the present-state of knowledge on the above model. It seems that only kernels of the form

$$\mu \,\equiv\, const \;, \quad \beta(y) \,=\, \beta\, y \;, \quad \kappa(y', y) \,=\, \frac{1}{y} \tag{5}$$

have been considered so far. This choice of kernels leads to a closed system of ordinary differential equations for v and

$$U(t) \,:=\, \int_{y_0}^{\infty} u(t, y)\, \mathrm{d}y \;, \quad P(t) := \int_{y_0}^{\infty} y\, u(t, y)\, \mathrm{d}y \;.$$

Indeed, (1) reduces to

$$\dot{v} \,=\, \lambda - \gamma\, v - \tau\, v\, U \,+\, \beta\, y_0^2\, U \;, \tag{6}$$

and integrating (2) yields the equations

$$\dot{U} \,=\, \beta\, P \,-\, \mu\, U \,-\, 2\, \beta\, y_0\, U \;, \tag{7}$$

$$\dot{P} \,=\, \tau\, v\, U \,-\, \mu\, P \,-\, \beta\, y_0^2\, U \;, \tag{8}$$

which, together with (6), are uniquely globally solvable. In addition, it has been shown in [5] that the disease-free steady state $(v, U, P) = (\lambda/\gamma, 0, 0)$ for the equations (6)-(8) is globally stable provided

$$\beta y_0 + \mu > \sqrt{\frac{\beta \lambda \tau}{\gamma}} \ . \tag{9}$$

If one reverses the strict inequality sign in (9) it has also been proved in [5] that there exists a prion disease steady state which is locally asymptotically stable. These results have been improved in [10] in that the disease-free steady state is globally asymptotically stable also for an equality sign in (9) and in that the disease steady state is even globally asymptotically stable for (9) with a reversed strict inequality sign.
Observe that the solvability of (6)-(8) implies that the original equations (1), (2) are no longer coupled since v is completely determined for all $t \geq 0$. Hence, as shown in [3], the partial differential equation (2) (with kernels as in (5)) can be solved for $u = u(t, y)$ by using the method of characteristics combined with semigroup theory. Moreover, it has also been shown in [3] that u converges either to 0 or to the disease steady state according to whether or not (9) holds.

Our aim is to consider quite general kernels, merely assuming suitable growth conditions. More precisely, after collecting some auxiliary results in section 2, we show in section 3 that (1)-(4) is globally well-posed provided μ and β are bounded, see Theorem 3.1. The basic idea is to solve equation (1) for a fixed, suitable function $\bar{u}$ and then to substitute the obtained solution $v_{\bar{u}}$ into equation (2). Using Kato's theory for hyperbolic evolution equations, we solve then equation (2) in order to obtain a classical solution $u_{\bar{u}}$. A fixed point argument for the map $\bar{u} \mapsto u_{\bar{u}}$ yields then local existence and uniqueness of a solution pair (v, u) for (1)-(4). Suitable a priori estimates guarantee global existence. A weak formulation of (2) allows then to extend in section 4 the existence results to unboundend kernels by using a weak compactness method, see Theorem 4.3. We also prove finite speed of propagation for the weak (and classical) solutions to (2). Finally, in section 5 we show that the disease-free steady state is globally asymptotically stable provided some suitable lower and upper bounds for the splitting kernels are available. We refer to Theorem 5.3 for a precise statement.
Clearly, the method described above does not yield uniqueness of weak solutions. This issue will be the topic of future work [8].

2. Preliminaries

In the following, we set $Y := (y_0, \infty)$ and assume that

$$\mu \, , \, \beta \in L_\infty^+(Y) \ , \tag{10}$$

where $L_\infty^+(Y)$ stands for the positive cone in $L_\infty(Y)$. We also assume that $\kappa \geq 0$ is measurable on $K := \{(y', y) \, ; \, y_0 < y < \infty \, , \, 0 < y' < y\}$ and satisfies

$$\kappa(y', y) = \kappa(y - y', y) \ , \quad (y', y) \in K \ , \tag{11}$$

which means binary splitting. Moreover, we suppose the number of monomer units to be preserved during splitting, that is,

$$2\int_0^y y'\,\kappa(y',y)\,\mathrm{d}y' \;=\; y\;, \quad \text{a.e. } y\in Y\;. \tag{12}$$

Furthermore, we let

$$\tau>0\;, \quad \lambda\,,\,\gamma\geq 0\;. \tag{13}$$

It is easy to check that (11), (12) imply

$$\int_0^y \kappa(y',y)\,\mathrm{d}y' \;=\; 1\;, \quad \text{a.e. } y\in Y\;. \tag{14}$$

Observe that the natural constraints (11), (12) hold if κ is of self-similar form

$$\kappa(y',y)\;=\;\frac{1}{y}\,\kappa_0\left(\frac{y'}{y}\right)\;, \quad y>y_0\;, \quad 0<y'<y\;, \tag{15}$$

where κ_0 is a non-negative integrable function defined on $(0,1)$ such that

$$\kappa_0(y)\;=\;\kappa_0(1-y)\;, \quad y\in(0,1)\;, \quad \int_0^1 \kappa_0(y)\;\mathrm{d}y\;=\;1\;. \tag{16}$$

This allows to capture κ in (5) by taking $\kappa_0\equiv 1$. Also note that the operator L, given by

$$L[u](y)\;:=\;-\big(\mu(y)\,+\,\beta(y)\big)\,u(y)\,+\,2\int_y^\infty \beta(y')\,\kappa(y,y')\,u(y')\,\mathrm{d}y'\;, \quad \text{a.e. } y\in Y\;, \tag{17}$$

defines a linear and bounded operator from $L_1(Y,y\mathrm{d}y)$ into itself according to (10)-(12) and that

$$\begin{aligned}\int_{y_0}^\infty \varphi(y)\,L[u](y)\,\mathrm{d}y \;=\; &-\int_{y_0}^\infty \varphi(y)\,\mu(y)\,u(y)\,\mathrm{d}y\\ &+\int_{y_0}^\infty u(y)\,\beta(y)\,\left(-\,\varphi(y)\,+\,2\int_{y_0}^y \varphi(y')\,\kappa(y',y)\,\mathrm{d}y'\right)\,\mathrm{d}y\end{aligned} \tag{18}$$

for $u\in L_1(Y,y\mathrm{d}y)$ and a suitable test function φ. We then put

$$E_0:=L_1(Y,y\mathrm{d}y) \quad \text{and} \quad E_1:=\mathring{W}_1^1(Y,y\mathrm{d}y):=cl_{W_1^1(Y,y\mathrm{d}y)}\mathcal{D}(Y)\;,$$

where $\mathcal{D}(Y)$ denotes the space of all test functions on Y. By E_0^+ we mean the positive cone in E_0 and $E_1^+:=E_1\cap E_0^+$. Finally, given any interval J and any function $v:J\to\mathbb{R}^+$, we define

$$\mathbb{A}_v(t)\,u\;:=\;\tau\,v(t)\,\partial_y u\,-\,L[u]\;, \quad u\in E_1\;, \quad t\in J\;. \tag{19}$$

Lemma 2.1. *The operator $-A$, defined as*

$$A\varphi:=\partial_y\varphi\;, \quad \varphi\in E_1\;, \tag{20}$$

generates a strongly continuous semigroup $\{\mathrm{e}^{-tA}\,;\,t\geq 0\}$ on E_0. It is given by

$$[e^{-tA}\varphi](y)=\begin{cases}\varphi(y-t)\;, & y>y_0+t\;,\\ 0\;, & y_0<y\leq y_0+t\;,\end{cases} \qquad t\geq 0\;, \tag{21}$$

and satisfies

$$\left\|e^{-tA}\right\|_{\mathcal{L}(E_0)}\leq e^{t/y_0}\;, \quad t\geq 0\;. \tag{22}$$

Proof. Clearly, (21) defines a strongly continuous semigroup on E_0 satisfying

$$\left\|e^{-tA}\varphi\right\|_{E_0} \le \left(1+\frac{t}{y_0}\right)\|\varphi\|_{E_0} \le e^{t/y_0}\|\varphi\|_{E_0} \ , \quad t\ge 0 \ ,$$

for $\varphi \in E_0$, whence (22). It thus remains to show that its generator $-A$ is indeed given by (20). Note that Lebesgue's theorem guarantees that the test functions are contained in the domain of A and that

$$A\varphi = \partial_y \varphi \ , \quad \varphi \in \mathcal{D}(Y) \ . \tag{23}$$

Since (21) is a right translation, $\mathcal{D}(Y)$ is invariant under e^{-tA} and therefore is a core for A. In particular, $\mathcal{D}(Y)$ is dense in the domain of A, which, together with (23), easily yields (20). □

In the sequel, we set $J_T := [0,T]$ for $T>0$ and, given $R>1$, we define

$$\mathcal{V}_{T,R} := \left\{v \in C^1(J_T)\, ;\, R^{-1} \le v(t) \le \|v\|_{C^1(J_T)} \le R\right\} \ . \tag{24}$$

Recall then that the operator $\mathbb{A}_v(t)$ has been defined in (19).

Proposition 2.2. *Fix $R>1$, $T_0>0$ and let $0<T\le T_0$. Then $\big(-\mathbb{A}_v(t)\big)_{t\in[0,T]}$ generates for each $v\in\mathcal{V}_{T,R}$ a unique evolution system $U_v(t,s)$, $0\le s\le t\le T$, in E_0, and there exists a constant $\omega_0 := \omega_0(T_0,R)>0$ such that*

$$\|U_v(t,s)\|_{\mathcal{L}(E_0)} \le e^{\omega_0(t-s)} \ , \quad 0\le s\le t\le T \ , \quad v\in\mathcal{V}_{T,R} \ , \tag{25}$$

and

$$\|U_v(t,s)\|_{\mathcal{L}(E_1)} \le \omega_0 \ , \quad 0\le s\le t\le T \ , \quad v\in\mathcal{V}_{T,R} \ . \tag{26}$$

Moreover, for $v,w\in\mathcal{V}_{T,R}$, it holds that

$$\|U_v(t,s)-U_w(t,s)\|_{\mathcal{L}(E_1,E_0)} \le \omega_0\,(t-s)\,\|v-w\|_{C(J_T)} \ , \quad 0\le s\le t\le T \ . \tag{27}$$

Proof. Since L is a bounded operator on E_0, Lemma 2.1 and a well-known perturbation result (see [9, Thm.3.1.1]) ensure that, for any fixed $v\in\mathcal{V}_{T,R}$ and any $s\in J_T$, $-\mathbb{A}_v(s)$ generates a strongly continuous semigroup on E_0 with

$$\|e^{-t\mathbb{A}_v(s)}\|_{\mathcal{L}(E_0)} \le e^{\bar\omega t} \ , \quad t\ge 0 \ , \tag{28}$$

where $\bar\omega := \tau R y_0^{-1} + \|L\|_{\mathcal{L}(E_0)}$. Hence, putting $\omega := \bar\omega + 1$ it follows that $\left\{\mathbb{A}_v(t)\right\}_{t\in J_T}$ is stable in the sense of [9, §5.2] for each $v\in\mathcal{V}_{T,R}$. Next, given any $t\in J_T$, the definition $Q_v(t) := \omega + \mathbb{A}_v(t)$ yields an isomorphism from E_1 onto E_0 satisfying

$$\|Q_v(t)\|_{\mathcal{L}(E_1,E_0)} \le \omega + \tau R + \|L\|_{\mathcal{L}(E_0)} \ , \quad t\in J_T \ , \quad v\in\mathcal{V}_{T,R} \ . \tag{29}$$

Moreover, for $u\in E_1$,

$$Q_v(\cdot)u \in C^1(J_T,E_0) \quad \text{with} \quad \dot Q_v(t)\,u \,:=\, \frac{\mathrm{d}}{\mathrm{d}t}\,Q_v(t)\,u \,=\, \tau\,\dot v(t)\,\partial_y u \ .$$

Therefore, assumptions $(H_1), (H_2)^+, (H_3)$ of [9, §5] hold, thus implying that there indeed exists a unique evolution system $U_v(t,s)$, $0\le s\le t\le T$, in E_0 for each $v\in\mathcal{V}_{T,R}$, which, in addition, satisfies statements $(E_1)-(E_5)$ of [9, §5]. In particular, (25) holds (with ω_0 replaced by $\bar\omega$).
We now refer to the proof of [9, Thm.5.4.6]: The evolution system $U_v(t,s)$ can be written as

$$U_v(t,s) \,=\, Q_v(t)^{-1}\,W_v(t,s)\,Q_v(s) \ , \quad 0\le s\le t\le T \ , \tag{30}$$

where $W_v(t,s) \in \mathcal{L}(E_0)$ satisfies

$$W_v(t,s)\,u \;=\; U_v(t,s)\,u \;+\; \int_s^t W_v(t,r)\,C_v(r)\,U_v(r,s)\,u\;\mathrm{d}r$$

for $0 \le s \le t \le T$ and $u \in E_0$ with

$$C_v(t) \;:=\; \dot{Q}_v(t)\,Q_v(t)^{-1} \in \mathcal{L}(E_0)\;, \quad t \in J_T\;.$$

We then claim that there is a constant $c_0(R) > 0$ such that

$$\|Q_v(t)^{-1}\|_{\mathcal{L}(E_0,E_1)} \le c_0(R)\;, \quad t \in J_T\;, \quad v \in \mathcal{V}_{T,R}\;. \tag{31}$$

Indeed, (28) implies

$$\|Q_v(t)^{-1}\|_{\mathcal{L}(E_0)} \le 1\;, \quad t \in J_T\;,$$

and therefore, for $u \in E_0$ and $t \in J_T$,

$$\begin{aligned}\|Q_v(t)^{-1}\,u\|_{E_1} \;&=\; \|Q_v(t)^{-1}\,u\|_{E_0} \;+\; \|\partial_y Q_v(t)^{-1}\,u\|_{E_0}\\ &\le\; \|u\|_{E_0} \;+\; \frac{1}{\tau v(t)}\,\|u \;-\; (\omega \;-\; L)\,Q_v(t)^{-1}\,u\|_{E_0}\\ &\le\; \big(1 \;+\; R/\tau\,(1 \;+\; \omega \;+\; \|L\|_{\mathcal{L}(E_0)})\big)\,\|u\|_{E_0}\;,\end{aligned}$$

whence (31). Consequently, we have

$$\begin{aligned}\|C_v(t)\|_{\mathcal{L}(E_0)} &\le \|\dot{Q}_v(t)\|_{\mathcal{L}(E_1,E_0)}\,\|Q_v(t)^{-1}\|_{\mathcal{L}(E_0,E_1)}\\ &\le \tau\,\|\dot{v}\|_{C(J_T)}\,c_0(R) \le c_0'(R)\end{aligned}$$

for $t \in J_T$ and $v \in \mathcal{V}_{T,R}$. From the proof of [9, Lem.5.4.5] (see in particular equation (4.11) therein) and from (25) it thus follows that there exists a constant $c(T_0,R) > 0$ such that

$$\|W_v(t,s)\|_{\mathcal{L}(E_0)} \le c(T_0,R)\;, \quad 0 \le s \le t \le T\;, \quad v \in \mathcal{V}_{T,R}\;. \tag{32}$$

Applying estimates (29), (31), and (32) to (30) we conclude that (26) is true.
Finally, let $v, w \in \mathcal{V}_{T,R}$ and $u \in E_1$ be arbitrary. Then, for $0 \le s < t \le T$,

$$N \;:=\; [\sigma \mapsto U_v(t,\sigma)\,U_w(\sigma,s)\,u] \in C^1((s,t),E_0) \cap C([s,t],E_1)$$

by $(E_2) - (E_5)$ in [9, §5] with

$$\dot{N}(\sigma) \;=\; U_v(t,\sigma)\,\big(\mathbb{A}_v(\sigma) - \mathbb{A}_w(\sigma)\big)\,U_w(\sigma,s)\,u\;.$$

Therefore, (25) and (26) yield

$$\begin{aligned}&\|U_w(t,s)u - U_v(t,s)u\|_{E_0}\\ &\qquad\le \int_s^t \|U_v(t,\sigma)\|_{\mathcal{L}(E_0)}\,\|\mathbb{A}_v(\sigma) - \mathbb{A}_w(\sigma)\|_{\mathcal{L}(E_1,E_0)}\,\|U_w(\sigma,s)\|_{\mathcal{L}(E_1)}\;\mathrm{d}\sigma\;\|u\|_{E_1}\\ &\qquad\le c(T_0,R)\,(t-s)\,\|v-w\|_{C(J_T)}\,\|u\|_{E_1}\end{aligned}$$

for $0 \le s \le t \le T$, hence statement (27). □

Remark 2.3. *As observed in the previous proof, the evolution system* $U_v(t,s)$, $0 \le s \le t \le T$, *corresponding to* $v \in \mathcal{V}_{T,R}$ *satisfies* $(E_1) - (E_5)$ *in* [9, §5]. *In particular, we have for* $u^0 \in E_1$ *that*

$$\big[t \mapsto U_v(t,0)\,u^0\big] \in C^1(J_T,E_0) \cap C(J_T,E_1)\;.$$

The existence of weak solutions will require the following auxiliary result.

Lemma 2.4. *For $v \in C(J_T)$ with $v(t) \geq 0$ put $A_v(t) := \tau v(t)\partial_y$, $t \in J_T$, and let $U_{A_v}(t,s)$, $0 \leq s \leq t \leq T$, be the corresponding evolution system in $L_1(Y)$. Then, for any $\delta > 0$, it holds that*

$$\sup_{|\mathcal{E}|\leq\delta} \int_{\mathcal{E}} U_{A_v}(t,s)\,\varphi\,\mathrm{d}y \leq \sup_{|\mathcal{E}|\leq\delta} \int_{\mathcal{E}} \varphi\,\mathrm{d}y\ , \quad 0 \leq s \leq t \leq T\ , \quad \varphi \in L_1^+(Y)\ ,$$

the supremum being taken over all measurable sets $\mathcal{E} \subset Y$.

Proof. Noticing that $-\partial_y$ with domain $\mathring{W}_1^1(Y)$ generates a strongly continuous positive semigroup of contractions on $L_1(Y)$ given as in (21), it follows that

$$\|e^{-tA_v(s)}\|_{\mathcal{L}(L_1(Y))} \leq 1\ , \quad \|e^{-tA_v(s)}\|_{\mathcal{L}(\mathring{W}_1^1(Y))} \leq 1\ , \quad t \geq 0\ , \quad s \in J_T\ .$$

Hence, the corresponding evolution system $U_{A_v}(t,s)$, $0 \leq s \leq t \leq T$, in $L_1(Y)$ is well-defined according to [9, Thm.5.2.2, Thm.5.3.1]. Let then $\mathcal{E} \subset Y$ be any measurable subset of Y with measure $|\mathcal{E}| \leq \delta$ and choose $\varphi \in L_1^+(Y)$. Denoting by χ_S the characteristic function on a set S, we have

$$\int_{\mathcal{E}} [e^{-tA_v(s)}\,\varphi](y)\,\mathrm{d}y = \int_{y_0}^{\infty} \chi_{\{-t\tau v(s)+\mathcal{E}\}}(y)\,\varphi(y)\,\mathrm{d}y \leq \sup_{|\mathcal{E}'|\leq\delta} \int_{\mathcal{E}'} \varphi(y)\,\mathrm{d}y$$

for $s \in J_T$ and $t \geq 0$. From equations (3.5) and (3.15) in [9, §5] we thus deduce

$$\int_{\mathcal{E}} U_{A_v}(t,s)\,\varphi\,\mathrm{d}y \leq \sup_{|\mathcal{E}'|\leq\delta} \int_{\mathcal{E}'} \varphi\,\mathrm{d}y\ , \quad 0 \leq s \leq t \leq T\ ,$$

and the assertion follows. □

3. Classical Solutions

In this section we show that problem (1)-(4) is globally well-posed for bounded kernels μ and β. In order to do this, let us denote by $|\cdot|_1$ the norm in $L_1(Y)$ and put

$$g(u) := 2\int_{y_0}^{\infty} u(y)\,\beta(y) \int_0^{y_0} y'\,\kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y\ .$$

Defining L by (17) and $\mathbb{A}_v(t)$ by (19), we may rewrite (1)-(4) as

$$\dot{v} = \lambda - \gamma v - \tau v\,|u|_1 + g(u)\ , \quad t > 0\ , \qquad v(0) = v^0\ , \tag{33}$$

provided $u \geq 0$, and

$$\dot{u} + \mathbb{A}_v(t)\,u = 0\ , \quad t > 0\ , \qquad u(0) = u^0\ . \tag{34}$$

Theorem 3.1. *Suppose* (10)-(13) *hold. Then, given any $v^0 > 0$ and $u^0 \in E_1^+$, problem* (33), (34) *possesses a unique global classical solution (v,u) such that $v \in C^1(\mathbb{R}^+)$, $v(t) > 0$ for $t > 0$, and $u \in C^1(\mathbb{R}^+, E_0) \cap C(\mathbb{R}^+, E_1^+)$.*

Proof. (i) We first prove that, for any $S > 0$, there exists $T := T(S) \in (0,1]$ such that (33), (34) possesses a unique solution (v,u) on J_T with regularity properties as stated in the theorem, provided that $(v^0,u^0) \in \mathbb{R}^+ \times E_1^+$ satisfy

$$S^{-1} \leq v^0 \quad \text{and} \quad v^0 + \|u^0\|_{E_1} \leq S\ . \tag{35}$$

In the following, we denote by $c(S) > 0$ a generic constant depending on S but not on $T \in (0,1]$. Let us then define the complete metric space

$$X_T := \big\{u \in C(J_T, E_0^+)\,;\, \|u(t)\|_{E_0} \leq S+1\,,\, t \in J_T\big\}\ ,$$

and let us choose $\bar u \in X_T$ arbitrarily. Then, since $g(\bar u), |\bar u|_1 \in C(J_T)$ due to (12), it follows that (33), with u replaced by $\bar u$, admits a unique solution $v_{\bar u} \in C^1(J_T)$. Clearly,

$$\begin{aligned} v_{\bar u}(t) \, &= \, e^{-\gamma t - \tau \int_0^t |\bar u(\sigma)|_1 \mathrm{d}\sigma}\, v^0 \\ &\quad + \int_0^t e^{-\gamma(t-s) - \tau \int_s^t |\bar u(\sigma)|_1 \mathrm{d}\sigma} \big(\lambda \,+\, g(\bar u(s))\big)\, \mathrm{d}s \end{aligned}$$

for $t \in J_T$, hence

$$v_{\bar u}(t) \, \ge \, e^{-\gamma t - \tau / y_0 (S+1) t}\, v^0 \, \ge \, c(S)\ , \quad 0 \le t \le T \le 1\ . \tag{36}$$

Moreover, since $v^0 \le S$ and $g(\bar u(t)) \le \|\beta\|_\infty (S+1)$ for $t \in J_T$, we deduce

$$v_{\bar u}(t) \, \le \, c(S)\ , \quad t \in J_T\ , \tag{37}$$

from which it follows

$$-c(S) \, \le \, -\big(\gamma \,+\, \tau\, |\bar u(t)|_1\big)\, v_{\bar u}(t) \, \le \, \dot v_{\bar u}(t) \, \le \, \lambda \,+\, g(\bar u(t)) \, \le \, c(S)\ , \quad t \in J_T\ . \tag{38}$$

Therefore, (36)-(38) entail the existence of $R := R(S) > 1$, depending on $S > 0$ but not on $T \in (0,1]$, such that $v_{\bar u} \in \mathcal{V}_{T,R}$ whenever $\bar u \in X_T$, where $\mathcal{V}_{T,R}$ is given by (24). Furthermore, we readily derive from the explicit representation of $v_{\bar u}$ and the linearity of g that

$$|v_{\bar u_1}(t) \,-\, v_{\bar u_2}(t)| \, \le \, c(S)\, \|\bar u_1 \,-\, \bar u_2\|_{X_T}\ , \quad 0 \le t \le T \le 1\ , \quad \bar u_1\, ,\, \bar u_2 \in X_T\ . \tag{39}$$

Let $U_{v_{\bar u}}(t,s)$, $0 \le s \le t \le T$, denote the unique evolution system in E_0 corresponding to $\big\{\mathbb{A}_{v_{\bar u}}(t)\big\}_{t \in J_T}$ and by $\omega_0 = \omega_0(1, R(S))$ the constant occurring in Proposition 2.2. Defining

$$\Lambda(\bar u)(t) \, := \, U_{v_{\bar u}}(t,0)\, u^0\ , \quad t \in J_T\ , \quad \bar u \in X_T\ ,$$

we obtain by Remark 2.3 the unique solution in $C(J_T, E_1) \cap C^1(J_T, E_0)$ to

$$\dot u \,+\, \mathbb{A}_{v_{\bar u}}(t)\, u \, = \, 0\ , \quad t > 0\ , \qquad u(0) \, = \, u^0\ .$$

Next we show that $\Lambda : X_T \to X_T$ is a contraction, which, consequently, would imply our first claim. Provided $T := T(S) \in (0,1]$ is chosen sufficiently small, we deduce from (25) that, for $\bar u \in X_T$ and $t \in J_T$,

$$\|\Lambda(\bar u)(t)\|_{E_0} \, \le \, e^{\omega_0 T}\, \|u^0\|_{E_0} \, \le \, S \,+\, 1\ ,$$

and (27) and (39) ensure for $\bar u_1, \bar u_2 \in X_T$ and $t \in J_T$

$$\|\Lambda(\bar u_1)(t) \,-\, \Lambda(\bar u_2)(t)\|_{E_0} \, \le \, \omega_0\, T\, \|v_{\bar u_1} \,-\, v_{\bar u_2}\|_{C(J_T)}\, \|u^0\|_{E_1} \, \le \, \frac{1}{2}\, \|\bar u_1 \,-\, \bar u_2\|_{X_T}\ .$$

In order to prove that $\Lambda(\bar u)(t)$ is non-negative observe that $\Lambda(\bar u)$ also solves

$$\dot u \,+\, \big(A_{v_{\bar u}}(t) \,+\, r\big)\, u \, = \, L[u] \,+\, r\, u \, =: \, B(u)\ , \quad t > 0\ , \qquad u(0) = u^0\ ,$$

with $r := \|\mu + \beta\|_\infty$ and $A_{v_{\bar u}}(t) := \tau v_{\bar u}(t) \partial_y$. Then $B(u) \in E_0^+$ for $u \in E_0^+$. Since Lemma 2.1 ensures that $-A_{v_{\bar u}}(s)$ generates a positive semigroup on E_0, it readily follows from the proof of [9, Thm.5.3.1] that the evolution system $\bar U(t,s)$ generated by $\big\{A_{v_{\bar u}}(t) + r\big\}_{t \in J_T}$ is positive. Defining then

$$F(w)(t) := \bar U(t,0)\, u^0 + \int_0^t \bar U(t,s)\, B(w(s))\, \mathrm{d}s$$

one shows that F is a contraction from a suitable closed ball in $C([0,\tilde{T}],E_0)$, containing u^0, into itself provided $\tilde{T}\in(0,T]$ is sufficiently small. Hence, putting

$$u_0 := u^0\ ,\quad u_{n+1} := F(u_n)\ ,\quad n\in\mathbb{N}\ ,$$

we obtain a sequence in $C([0,\tilde{T}],E_0^+)$ that converges to $\Lambda(\bar{u})|_{[0,\tilde{T}]}$. This shows that

$$T^\star := \sup\{T'\in(0,T]\,;\,\Lambda(\bar{u})(t)\in E_0^+\ ,\ 0\le t\le T'\}\ \ge\ \tilde{T}\ .$$

Assuming $T^\star < T$, a repetition of the above arguments with u^0 replaced by $\Lambda(\bar{u})(T^\star)\in E_1^+$ would lead to a contradiction. Thus $T^\star = T$, which entails that $\Lambda : X_T\to X_T$ is indeed a contraction.

(ii) It follows from part (i) that (33), (34) admits a unique maximal solution

$$(v,u)\in C(J,\mathbb{R}^+\times E_1^+)\cap C^1(J,\mathbb{R}\times E_0)\ ,$$

where J is open in $\mathbb{R}^+$. We claim that, if $t^+ := \sup J < \infty$, then

$$\varliminf_{t\nearrow t^+} v(t) = 0 \qquad \text{or} \qquad \varlimsup_{t\nearrow t^+}\big(v(t)+\|u(t)\|_{E_1}\big) = \infty\ . \tag{40}$$

For, suppose to the contrary that there are $t_j\nearrow t^+<\infty$ and $S>0$ such that

$$v(t_j)\ge S^{-1} \qquad \text{and} \qquad v(t_j)+\|u(t_j)\|_{E_1}\le S\ .$$

Let $T(S)>0$ be the corresponding constant from part (i) and fix $t_N > t^+ - T(S)$. Then we may choose $\big(v(t_N),u(t_N)\big)\in\mathbb{R}^+\setminus\{0\}\times E_1^+$ as initial value and deduce that the solution (v,u) can be extended to a solution on $[0,t_N+T(S)]$, contradicting its maximality.

(iii) We now show that (40) does not occur in finite time. Observe that (12) and (18) imply

$$\dot{v}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\int_{y_0}^\infty y\,u(t,y)\,\mathrm{d}y = \lambda - \gamma\,v(t) - \int_{y_0}^\infty y\,\mu(y)\,u(t,y)\,\mathrm{d}y\ ,\quad t\in J\ , \tag{41}$$

hence

$$v(t) + \|u(t)\|_{E_0} \le v^0 + \|u^0\|_{E_0} + \lambda t\ ,\quad t\in J\ . \tag{42}$$

Suppose now that $t^+<\infty$. Then (42) entails that

$$\dot{v}(t) \le \lambda + g(u(t)) \le \lambda + \|\beta\|_\infty\,\|u(t)\|_{E_0} \le c(t^+)\ ,\quad t\in J\ ,$$

and

$$\dot{v}(t) \ge -\gamma\,v(t) - \tau\,|u(t)|_1\,v(t) \ge -c(t^+)\ ,\quad t\in J\ .$$

Therefore

$$\|v\|_{C^1(J)}\le c(t^+) \tag{43}$$

and

$$v(t)\ge e^{-(\gamma+\tau|u(t)|_1)t}\,v^0 \ge e^{-(\gamma+\tau c(t^+))t^+}\,v^0 > 0\ ,\quad t\in J\ . \tag{44}$$

Taking (26) into account, we derive from (43), (44) that the evolution system $U_v(t,s)$ satisfies

$$\|U_v(t,s)\|_{\mathcal{L}(E_1)}\le c(t^+)\ ,\quad 0\le s\le t<t^+\ .$$

But then

$$\|u(t)\|_{E_1} = \|U_v(t,0)u^0\|_{E_1} \le c(t^+)\,\|u^0\|_{E_1}\ ,\quad t\in J\ , \tag{45}$$

thus (40) cannot be true in view of (43) - (45). This contradiction proves that the solution (v,u) exists for all times, hence the assertion follows. □

If (v,u) denotes the solution to (1)-(4) provided by Theorem 3.1, the next proposition shows that u propagates with finite speed. The proof is adapted from the proof of [6, Thm.2.6].

Proposition 3.2. *Suppose* (10)-(13) *hold. For* $v^0>0$ *and* $u^0\in E_1^+$ *let* (v,u) *denote the unique global classical solution to* (1)-(4). *If* $\operatorname{supp} u^0\subset[y_0,S_0]$ *for some* $S_0>y_0$, *then* $\operatorname{supp} u(t)\subset[y_0,S(t)]$, $t\ge 0$, *where*

$$S(t):=S_0+\tau\int_0^t v(s)\,\mathrm{d}s\ ,\quad t\ge 0\ .$$

Proof. Define $P\in C^1(\mathbb{R}^+,L_1(Y))$ by

$$P(t,y):=\int_y^\infty u(t,y')\,\mathrm{d}y'\ ,\quad y\in Y\ ,\quad t\ge 0\ .$$

Then, since

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t,y)=\int_y^\infty \dot u(t,y')\,\mathrm{d}y'=\tau\,v(t)\,u(t,y)+\int_y^\infty L[u(t)](y')\,\mathrm{d}y'\ ,$$

we derive from (2) and (14)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S(t)}^\infty P(t,y)\,\mathrm{d}y&=\int_{S(t)}^\infty\frac{\mathrm{d}}{\mathrm{d}t}P(t,y)\,\mathrm{d}y\;-\;S'(t)\,P(t,S(t))\\&=\int_{S(t)}^\infty\int_y^\infty L[u(t)](y')\,\mathrm{d}y'\,\mathrm{d}y\\&\le 2\int_{S(t)}^\infty\int_y^\infty\int_{y'}^\infty\beta(y'')\,\kappa(y',y'')\,u(t,y'')\,\mathrm{d}y''\,\mathrm{d}y'\,\mathrm{d}y\\&=2\int_{S(t)}^\infty\int_y^\infty\beta(y'')\,u(t,y'')\int_y^{y''}\kappa(y',y'')\,\mathrm{d}y'\,\mathrm{d}y''\,\mathrm{d}y\\&\le 2\,\|\beta\|_\infty\int_{S(t)}^\infty P(t,y)\,\mathrm{d}y\ ,\end{aligned}$$

which implies

$$\int_{S(t)}^\infty P(t,y)\,\mathrm{d}y\;\le\;e^{2\|\beta\|_\infty t}\int_{S_0}^\infty\int_y^\infty u^0(y')\,\mathrm{d}y'\,\mathrm{d}y\;=\;0\ ,\quad t\ge 0\ .$$

Hence $u(t,y)=0$ for $y\in(S(t),\infty)$ and $t\ge 0$. □

Remark 3.3. *Note that if* $\mu(y)\ge\underline{\mu}>0$ *for a.e.* $y\in Y$ *and* $\gamma>0$, *then* (41) *entails*

$$v(t)\;+\int_{y_0}^\infty y\,u(t,y)\,\mathrm{d}y\;\le\;\frac{\lambda}{\nu}\;+\;e^{-\nu t}\left(v^0\;+\;\|u^0\|_{E_0}\;-\;\frac{\lambda}{\nu}\right)\ ,\quad t\ge 0\ ,\qquad(46)$$

where $\nu:=\min\{\underline{\mu},\gamma\}>0$. *In particular,*

$$\int_0^t v(s)\,\mathrm{d}s\;\le\;\frac{\lambda\,t}{\nu}\;+\;\frac{1}{\nu}\left(1\;-\;e^{-\nu t}\right)\left(v^0\;+\;\|u^0\|_{E_0}\;-\;\frac{\lambda}{\nu}\right)\ ,\quad t\ge 0\ .\qquad(47)$$

4. Weak Solutions

The aim of this section is to relax condition (10) and to prove existence of weak solutions for unbounded kernels μ and β. More precisely, instead of (10) we assume in the following that

$$\begin{cases} \text{there exists } \alpha \geq 1 \text{ and } \varrho \in L_\infty^+(Y) \text{ such that} \\ \varrho(y) \to 0 \text{ as } y \to \infty \text{ and } \mu(y) + \beta(y) \leq \varrho(y) y^\alpha, \text{ a.e. } y \in Y\ . \end{cases} \tag{48}$$

In addition, we require that

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \displaystyle\sup_{|\mathcal{E}| \leq \delta} \frac{\beta(y)}{y^\alpha} \int_{y_0}^{y} \chi_{\mathcal{E}}(y')\, \kappa(y', y)\, \mathrm{d}y' \ \leq\ \varepsilon\ , \quad \text{a.e. } y \in Y\ , \end{cases} \tag{49}$$

the supremum being taken over all measurable subsets $\mathcal{E}$ in Y with measure $|\mathcal{E}| \leq \delta$. Observe that if κ is subject to the self-similar form (15), (16), then

$$\lim_{|\mathcal{E}| \to 0} \operatorname*{ess\text{-}sup}_{y > y_0} \int_{y_0}^{y} \mathbf{1}_{\mathcal{E}}(y')\, \kappa(y', y)\ \mathrm{d}y' \ =\ \lim_{|\mathcal{E}| \to 0} \operatorname*{ess\text{-}sup}_{y > y_0} \int_{y_0/y}^{1} \mathbf{1}_{\frac{1}{y}\mathcal{E}}(y')\, \kappa_0(y')\ \mathrm{d}y' \ =\ 0$$

due to $y_0 > 0$ and the integrability of κ_0, so (49) automatically holds by (48).
In the following we denote by $L_{1,\mathsf{w}}(Y)$ the space $L_1(Y)$ equipped with its weak topology.

Definition 4.1. *Given $v^0 > 0$ and $u^0 \in L_1^+(Y, y\mathrm{d}y)$, we call (v, u) a (global) weak solution to* (1)-(4) *if*

(i) $g(u) \in C(\mathbb{R}^+)$,
(ii) $v \in C^1(\mathbb{R}^+)$ *is a non-negative solution to* (1),
(iii) $u \in C(\mathbb{R}^+, L_{1,\mathsf{w}}(Y)) \cap L_{\infty,loc}(\mathbb{R}^+, L_1^+(Y, y\mathrm{d}y))$,
(iv) *for all $t > 0$ and $\varphi \in W_\infty^1(Y)$ it holds that $L[u] \in L_1((0,t) \times Y)$ and*

$$\int_{y_0}^{\infty} \varphi(y)\, u(t,y)\, \mathrm{d}y - \tau \int_0^t v(s) \int_{y_0}^{\infty} \varphi'(y)\, u(s,y)\, \mathrm{d}y\, \mathrm{d}s$$
$$= \int_{y_0}^{\infty} \varphi(y)\, u^0(y)\, \mathrm{d}y + \int_0^t \int_{y_0}^{\infty} \varphi(y)\, L[u(s)](y)\, \mathrm{d}y\, \mathrm{d}s\ .$$

We first need the following auxiliary result.

Lemma 4.2. *Suppose that h_n and h are measurable functions on Y such that $h_n \to h$ a.e. and let $u_n \to u$ in $L_{1,\mathsf{w}}^+(Y)$.*

(i) *If $\|h_n\|_\infty \leq c$, then $h_n u_n \to hu$ in $L_{1,\mathsf{w}}(Y)$.*
(ii) *If ϱ and α are as in* (48) *and if $|h_n(y)| \leq \varrho(y) y^\alpha$ for a.e. $y \in Y$ and*

$$\int_{y_0}^{\infty} y^\alpha\, u_n(y)\, \mathrm{d}y \leq c\ , \quad n \in \mathbb{N}\ ,$$

then $h_n u_n \to hu$ in $L_{1,\mathsf{w}}(Y)$.

Proof. In case that Y is a finite interval, a proof of (i) is implicitly contained in [11, Lem.4.1] (a detailed proof can also be found in [13, App.]). The case of unbounded Y is a slight modification thereof. Statement (ii) can be shown along the lines of

[7, App.A, Cor.4.1]. Nevertheless, for the reader's convenience, we include here a proof. First note that the assumptions imply $|h(y)| \le \varrho(y) y^\alpha$, a.e. $y \in Y$, and

$$\int_{y_0}^\infty y^\alpha\, u(y)\, \mathrm{d}y \le c .$$

Putting $\bar u_n(y) := \varrho(y) y^\alpha u_n(y)$ and $\bar u(y) := \varrho(y) y^\alpha u(y)$ we obtain for $\varphi \in L_\infty(Y)$ and $R > y_0$

$$\Big| \int_{y_0}^\infty \varphi(y) \big(\bar u_n(y) - \bar u(y)\big)\, \mathrm{d}y \Big| \le \Big| \int_{y_0}^R \varphi(y)\, \varrho(y)\, y^\alpha \big(u_n(y) - u(y)\big)\, \mathrm{d}y \Big| + 2\, c\, \|\varphi\|_\infty\, \|\varrho\|_{L_\infty(R,\infty)} .$$

Taking first the lim sup as $n \to \infty$ on both sides and letting then $R \to \infty$, we conclude from (48) that $\bar u_n \to \bar u$ in $L_{1,\mathsf{w}}(Y)$. Therefore, it follows from (i) that the right hand side of the estimate

$$\begin{aligned} &\Big| \int_{y_0}^\infty \varphi(y) \big(h_n(y)\, u_n(y) - h(y)\, u(y)\big)\, \mathrm{d}y \Big| \\ &\qquad \le \Big| \int_{y_0}^\infty \varphi(y) \big(\varrho(y)\, y^\alpha\big)^{-1} \big(h_n(y) - h(y)\big)\, \bar u_n(y)\, \mathrm{d}y \Big| \\ &\qquad\quad + \Big| \int_{y_0}^\infty \varphi(y) \big(\varrho(y)\, y^\alpha\big)^{-1} h(y) \big(\bar u_n(y) - \bar u(y)\big)\, \mathrm{d}y \Big| \end{aligned}$$

converges to 0, leading to the assertion. □

Now we are in a position to relax the boundedness assumptions on μ and β and also the assumption on u^0 can be weakened.

Theorem 4.3. *Suppose that* (11)-(13) *and* (48), (49) *hold. Then, given any* $v^0 > 0$ *and* $u^0 \in L_1^+(Y, y^\alpha \mathrm{d}y)$, *problem* (1)-(4) *admits at least one global weak solution* (v, u). *In addition,* u *belongs to* $L_{\infty,loc}(\mathbb{R}^+, L_1(Y, y^\alpha \mathrm{d}y))$.

Proof. (i) Let $u_n^0 \in \mathcal{D}^+(Y)$ be such that $u_n^0 \to u^0$ in $L_1(Y, y^\alpha \mathrm{d}y)$. We define $\mu_n := \min\{\mu, n\}$ and $\beta_n := \min\{\beta, n\}$. Observe that μ_n, β_n also satisfy (48) and (49). Then Theorem 3.1 guarantees the existence of

$$(v_n, u_n) \in C(\mathbb{R}^+, \mathbb{R}^+ \times E_1^+) \cap C^1(\mathbb{R}^+, \mathbb{R} \times E_0)$$

satisfying

$$\dot v_n = \lambda - \gamma\, v_n - \tau\, v_n\, |u_n|_1 + g_n(u_n) , \quad t > 0 , \qquad v_n(0) = v^0 , \tag{50}$$

and

$$\partial_t u_n + \tau\, v_n(t)\, \partial_y u_n = L_n[u_n] , \quad t > 0 , \quad u_n(0) = u_n^0 , \tag{51}$$

where

$$g_n(u) := 2 \int_{y_0}^\infty u(y)\, \beta_n(y) \int_0^{y_0} y'\, \kappa(y', y)\, \mathrm{d}y'\, \mathrm{d}y$$

and

$$L_n[u](y) := -\big(\mu_n(y) + \beta_n(y)\big)\, u(y) + 2 \int_y^\infty \beta_n(y')\, \kappa(y, y')\, u(y')\, \mathrm{d}y' .$$

Let $T > 0$ be arbitrary. According to (42) there exists $c_0(T) > 0$ independent of $n \ge 1$ such that

$$v_n(t) + \|u_n(t)\|_{E_0} \le c_0(T) , \quad t \in J_T , \quad n \ge 1 . \tag{52}$$

Moreover, we claim that

$$\|u_n(t)\|_{L_1(Y,y^\alpha \mathrm{d}y)} \le c_0(T) \ , \quad t \in J_T \ , \quad n \ge 1 \ . \tag{53}$$

For, recall that $u_n(t)$ has compact support due to Proposition 3.2. Hence, we may test (51) by y^α and obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{y_0}^\infty y^\alpha\, u_n(t,y)\,\mathrm{d}y \;&=\; \alpha\,\tau\, v_n(t) \int_{y_0}^\infty y^{\alpha-1}\, u_n(t,y)\,\mathrm{d}y \\ &\quad - \int_{y_0}^\infty y^\alpha \big(\mu_n(y)+\beta_n(y)\big)\, u_n(t,y)\,\mathrm{d}y \\ &\quad + 2\int_{y_0}^\infty u_n(t,y)\,\beta_n(y) \int_{y_0}^y (y')^\alpha\, \kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y \\ &\le\; \alpha\,\tau\, v_n(t) \int_{y_0}^\infty y^{\alpha-1}\, u_n(t,y)\,\mathrm{d}y \end{aligned}$$

for $t \ge 0$, since (12) ensures

$$2\int_{y_0}^y (y')^\alpha\, \kappa(y',y)\,\mathrm{d}y' \;\le\; y^\alpha \ , \quad \text{a.e. } y > y_0 \ .$$

Therefore, Gronwall's inequality and estimate (52) yield (53). In particular, combining (53), (48) and (14) we deduce

$$g_n(u_n(t)) \;\le\; 2\,y_0\, \|\varrho\|_\infty\, \|u_n(t)\|_{L_1(Y,y^\alpha \mathrm{d}y)} \;\le\; c(T) \ , \quad t \in J_T \ , \quad n \ge 1 \ .$$

(ii) It follows from (1) and the estimate on $g_n(u_n(t))$ that

$$|v_n(t) - v_n(s)| \;\le\; c(T)\, |t-s| \ , \quad t,s \in J_T \ , \quad n \ge 1 \ ,$$

where $c(T) > 0$ is independent of $n \ge 1$. Taking (52) into account, the Arzelà-Ascoli theorem warrants that the sequence (v_n) is relatively compact in $C(J_T)$.

(iii) We show that (u_n) is relatively sequentially compact in $C(J_T, L_{1,\mathsf{w}}(Y))$. According to a variant of the Arzelà-Ascoli theorem (see [12, Thm.1.3.2]) we merely have to check that the set $\{u_n(t)\,;\, n \ge 1\}$ is relatively compact in $L_{1,\mathsf{w}}(Y)$ for every $t \in J_T$ and that the set $\{u_n\,;\, n \ge 1\}$ is equicontinuous in $L_{1,\mathsf{w}}(Y)$ at every $t \in J_T$. First observe that (52) entails

$$\lim_{R\to\infty} \sup_{\substack{n\ge 1 \\ t\in J_T}} \int_R^\infty u_n(t,y)\,\mathrm{d}y \;=\; 0 \ . \tag{54}$$

Let $U_{v_n}(t,s)$ denote the evolution system in $L_1(Y)$ corresponding to the operator $A_{v_n}(t) := \tau v_n(t)\partial_y$. Then

$$u_n(t) \;=\; U_{v_n}(t,0)\, u_n^0 + \int_0^t U_{v_n}(t,s)\, L_n[u_n(s)]\,\mathrm{d}s \ , \quad t \in J_T \ .$$

Consequently, given $\delta > 0$, Lemma 2.4 and the positivity of $u_n(t)$ imply that

$$\begin{aligned} \sup_{|\mathcal{E}|\le\delta} \int_{\mathcal{E}} u_n(t,y)\,\mathrm{d}y \;\le\; & \sup_{|\mathcal{E}|\le\delta} \int_{\mathcal{E}} u_n^0(y)\,\mathrm{d}y \\ & + 2\int_0^t \sup_{|\mathcal{E}|\le\delta} \int_{y_0}^\infty u_n(s,y)\,\beta_n(y) \int_{y_0}^y \chi_{\mathcal{E}}(y')\,\kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \ . \end{aligned}$$

Since $u_n^0 \to u^0$ in $L_1(Y, y^\alpha \mathrm{d}y)$ and in view of (49) and (53), we conclude that

$$\lim_{|\mathcal{E}|\to 0} \sup_{\substack{n\ge 1\\ t\in J_T}} \int_{\mathcal{E}} u_n(t,y)\,\mathrm{d}y \;=\; 0\;. \tag{55}$$

From (52), (54), (55) and the Dunford-Pettis theorem (cf. [2, Thm.4.21.2]) we hence derive that $\{u_n(t)\,;\, t\in J_T\,, n\ge 1\}$ is relatively compact in $L_{1,\mathsf{w}}(Y)$.
Now let $\varphi \in \mathcal{D}(Y)$ be arbitrary. Testing (51) by φ, we infer

$$\begin{aligned}
\Big| \int_{y_0}^\infty & \varphi(y)\,[u_n(t,y) - u_n(s,y)]\,\mathrm{d}y\Big| \\
&\le \tau \int_s^t v_n(\sigma) \int_{y_0}^\infty |\varphi'(y)|\, u_n(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\
&\quad + \int_s^t \int_{y_0}^\infty |\varphi(y)|\,\big(\mu_n(y)+\beta_n(y)\big)\,u_n(\sigma,y)\,\mathrm{d}y\,\mathrm{d}\sigma \\
&\quad + 2\int_s^t \int_{y_0}^\infty u_n(\sigma,y)\,\beta_n(y) \int_{y_0}^y |\varphi(y')|\,\kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}\sigma
\end{aligned}$$

for $0\le s\le t\le T$, whence, from (14), (48), (52) and (53),

$$\Big|\int_{y_0}^\infty \varphi(y)[u_n(t,y)-u_n(s,y)]\,\mathrm{d}y\Big| \;\le\; c(T,\varphi)\,|t-s|\;,\quad t,s\in J_T\;. \tag{56}$$

For $\varphi\in L_\infty(Y)$ let $\varphi_j\in\mathcal{D}(Y)$ be such that $\varphi_j\to\varphi$ a.e. and $\|\varphi_j\|_\infty \le \|\varphi\|_\infty$ (see [1, p.131f]). Given $\varepsilon>0$ it follows from (54), from the fact that $\{u_n(t)\,;\, t\in J_T\,, n\ge 1\}$ is relatively compact in $L_{1,\mathsf{w}}(Y)$, and from Egorov's theorem that there are $R>y_0$, a measurable subset $\mathcal{E}$ of (y_0,R) and $j\in\mathbb{N}$ such that

$$\int_R^\infty u_n(t,y)\,\mathrm{d}y + \int_{\mathcal{E}} u_n(t,y)\,\mathrm{d}y \;\le\; \frac{\varepsilon}{12\,\|\varphi\|_\infty}\;,\quad t\in J_T\;,\quad n\ge 1\;,$$

and

$$\|\varphi-\varphi_j\|_{L_\infty((y_0,R)\setminus\mathcal{E})} \;\le\; \frac{\varepsilon}{6\,c_0(T)}\;,$$

where $c_0(T)>0$ stems from (52). Therefore, (56) yields

$$\begin{aligned}
\Big|\int_{y_0}^\infty \varphi(y)\,[u_n(t,y)-u_n(s,y)]\,\mathrm{d}y\Big| \;&\le\; \|\varphi-\varphi_j\|_{L_\infty((y_0,R)\setminus\mathcal{E})}\,\big(|u_n(t)|_1+|u_n(s)|_1\big) \\
&\quad+\;\big(\|\varphi\|_\infty+\|\varphi_j\|_\infty\big)\int_{\mathcal{E}}\big(u_n(t,y)+u_n(s,y)\big)\,\mathrm{d}y \\
&\quad+\;\big(\|\varphi\|_\infty+\|\varphi_j\|_\infty\big)\int_R^\infty\big(u_n(t,y)+u_n(s,y)\big)\,\mathrm{d}y \\
&\quad+\;c(T,\varphi_j)\,|t-s| \\
&\le\;\varepsilon\;+\;c(T,\varphi_j)\,|t-s|
\end{aligned}$$

for $t,s\in J_T$ and $n\ge 1$. We conclude

$$\lim_{s\to t}\sup_{n\ge 1}\Big|\int_{y_0}^\infty \varphi(y)\,[u_n(t,y)-u_n(s,y)]\,\mathrm{d}y\Big| \;=\; 0\;,$$

hence $\{u_n\,;\, n\ge 1\}$ is equicontinuous in $L_{1,\mathsf{w}}(Y)$ at every $t\in J_T$.

(iv) Since now (v_n, u_n) is relatively weakly compact in $C(J_T, \mathbb{R} \times L_{1,\mathsf{w}}(Y))$ for each $T > 0$, we may choose a subsequence (again denoted by $((v_n, u_n))_{n\in\mathbb{N}}$) and a function $(v, u) \in C(\mathbb{R}^+, \mathbb{R} \times L_{1,\mathsf{w}}(Y))$ such that

$$(v_n, u_n) \to (v, u) \quad \text{in} \quad C(J_T, \mathbb{R} \times L_{1,\mathsf{w}}(Y)) \tag{57}$$

for each $T > 0$.

(v) We then claim that (v, u) is a weak solution to (1)-(4). Evidently, it holds that $(v(t), u(t)) \in \mathbb{R}^+ \times L_1^+(Y)$ for $t > 0$ since $(v_n(t), u_n(t)) \in \mathbb{R}^+ \times L_1^+(Y)$. We fix again $T > 0$. Then (57) and (53) imply

$$\|u(t)\|_{L_1(Y, y^\alpha \mathrm{d}y)} \le c_0(T) \ , \quad t \in J_T \ , \tag{58}$$

in particular, we have $u \in L_{\infty,loc}(\mathbb{R}^+ L_1(Y, y^\alpha \mathrm{d}y))$. Let $\varphi \in W_\infty^1(Y)$ be arbitrary. Clearly, (57) yields

$$\lim_{n\to\infty} \int_{y_0}^\infty \varphi(y)\, u_n(t,y)\,\mathrm{d}y = \int_{y_0}^\infty \varphi(y)\, u(t,y)\,\mathrm{d}y \ , \quad t \in J_T \ . \tag{59}$$

Moreover, writing

$$\begin{aligned}\Big| \int_0^t v(s) \int_{y_0}^\infty \varphi'(y)\, u(s,y)\,\mathrm{d}y\,\mathrm{d}s - \int_0^t v_n(s) \int_{y_0}^\infty \varphi'(y)\, u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s \Big| \\ \le \int_0^t |v(s) - v_n(s)| \int_{y_0}^\infty |\varphi'(y)|\, u(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ + \int_0^t v_n(s) \Big| \int_{y_0}^\infty \varphi'(y)\,[\,u(s,y) - u_n(s,y)]\,\mathrm{d}y \Big|\,\mathrm{d}s\end{aligned}$$

for $t \in J_T$, we infer from (57), (52) and Lebesgue's theorem that, for $t \in J_T$,

$$\lim_{n\to\infty} \int_0^t v_n(s) \int_{y_0}^\infty \varphi'(y)\, u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s = \int_0^t v(s) \int_{y_0}^\infty \varphi'(y)\, u(s,y)\,\mathrm{d}y\,\mathrm{d}s \ . \tag{60}$$

In addition, since $\mu_n(y) + \beta_n(y) \le \varrho(y) y^\alpha$ for a.e. $y \in Y$, we conclude from Lemma 4.2(ii), (53), (57) and Lebesgue's theorem that

$$\begin{aligned}\lim_{n\to\infty} \int_0^t \int_{y_0}^\infty \varphi(y)\,\big(\mu_n(y) + \beta_n(y)\big)\, u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ = \int_0^t \int_{y_0}^\infty \varphi(y)\,\big(\mu(y) + \beta(y)\big)\, u(s,y)\,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

as well as

$$\begin{aligned}\lim_{n\to\infty} \int_0^t \int_{y_0}^\infty \varphi(y) \int_y^\infty u_n(s,y')\,\beta_n(y')\,\kappa(y,y')\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \\ = \int_0^t \int_{y_0}^\infty \varphi(y) \int_y^\infty u(s,y')\,\beta(y')\,\kappa(y,y')\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \ ,\end{aligned}$$

where we use Fubini's theorem for the second limit. Therefore,

$$\lim_{n\to\infty} \int_0^t \int_{y_0}^\infty \varphi(y)\, L_n[u_n(s)]\,\mathrm{d}y\,\mathrm{d}s = \int_0^t \int_{y_0}^\infty \varphi(y)\, L[u(s)]\,\mathrm{d}y\,\mathrm{d}s \ . \tag{61}$$

Now, since (v_n, u_n) is a weak solution to (1)-(4), we derive from (59)-(61) that u indeed satisfies part (iv) of Definition 4.1. Next, it follows from Lemma 4.2(ii), similarly as above, that

$$\lim_{n\to\infty} g_n(u_n(t)) \;=\; g(u(t)) \ , \quad t \in J_T \ ,$$

and also

$$\lim_{n\to\infty} \int_0^t |u_n(s)|_1 \,\mathrm{d}s \;=\; \int_0^t |u(s)|_1 \,\mathrm{d}s \ , \quad t \in J_T \ .$$

Consequently, (50) yields

$$\begin{aligned} v(t) \;&=\; e^{-\gamma t - \tau \int_0^t |u(\sigma)|_1 \mathrm{d}\sigma}\, v^0 \\ &\qquad + \int_0^t e^{-\gamma(t-s) - \tau \int_s^t |u(\sigma)|_1 \mathrm{d}\sigma} \big(\lambda + g(u(s))\big) \,\mathrm{d}s \end{aligned}$$

for $t \in J_T$. But since $u \in C(\mathbb{R}^+, L_{1,\mathsf{w}}(Y))$, Lemma 4.2(ii) and (58) warrant that $g(u) \in C(J_T)$. In addition, $|u|_1 \in C(J_T)$, so we deduce that $v \in C^1(J_T)$ solves (1). This proves the theorem. □

Also the weak solution propagates with finite speed as shown in the next corollary.

Corollary 4.4. *Suppose* (11)-(13), (48), (49). *If* $v^0 > 0$ *and if* $u^0 \in L_1^+(Y, y^\alpha \mathrm{d}y)$ *is such that* $\operatorname{supp} u^0 \subset [y_0, S_0]$, *then the weak solution* (v,u) *provided by Theorem 4.3 satisfies* $\operatorname{supp} u(t) \subset [y_0, S(t)]$ *for* $t \geq 0$, *where*

$$S(t) := S_0 + \tau \int_0^t v(s) \,\mathrm{d}s \ , \quad t \geq 0 \ .$$

Proof. We may choose the sequence $(u_n^0) \subset \mathcal{D}^+(Y)$ in the proof of Theorem 4.3 such that $\operatorname{supp} u_n^0 \subset (y_0, S_0)$. Then Proposition 3.2 ensures that the approximating sequence $((v_n, u_n))_{n\in\mathbb{N}}$ given in (50), (51) satisfies $\operatorname{supp} u_n(t) \subset [y_0, S_n(t)]$ for $t \geq 0$, where

$$S_n(t) \;:=\; S_0 \;+\; \tau \int_0^t v_n(s) \,\mathrm{d}s \ , \quad t \geq 0 \ , \quad n \geq 1.$$

Evidently, $\lim_{n\to\infty} S_n(t) = S(t)$ and

$$\int_{S(t)}^\infty u(t,y) \,\mathrm{d}y \;=\; \lim_{n\to\infty} \int_{S_n(t)}^\infty u_n(t,y) \,\mathrm{d}y \;=\; 0$$

by (57) and Lemma 4.2(i), thus $\operatorname{supp} u(t) \subset [y_0, S(t)]$ for $t \geq 0$. □

Remark 4.5. *In addition to* (11)-(13), (48), (49) *suppose that* $\mu(y) \geq \underline{\mu} > 0$ *for a.e.* $y \in Y$ *and that* $\gamma > 0$. *Then the weak solution* (v,u) *also satisfies the estimates* (46) *and* (47). *Indeed,* (46) *follows immediately from the corresponding estimate for* (v_n, u_n) *and* (57).

5. Stability of the Disease Free Steady State

This section is devoted to the investigation of stability properties of the disease-free steady state $(v, u) = (\lambda/\gamma, 0)$ of (1),(2).

In the sequel, we always assume that (11)-(13) are satisfied with $\gamma > 0$ and that either

$$\begin{cases} \text{(10) holds ,} \\ v^0 > 0 \ , \quad u^0 \in E_1^+ \ , \end{cases} \tag{62}$$

or

$$\begin{cases} \text{(48), (49) hold ,} \\ v^0 > 0 \ , \quad u^0 \in L_1^+(Y, y^\alpha \mathrm{d}y) \ . \end{cases} \tag{63}$$

Then we denote by (v, u) either the classical solution provided by Theorem 3.1 if (62) holds, or the weak solution provided by Theorem 4.3 if (63) holds.
We assume that

$$d_0 := \underset{y \in Y}{\text{ess-sup}} \, \frac{\beta(y)}{y\mu(y)} \in (0, \infty)$$

and introduce ε_k, δ_k such that

$$0 \leq \delta_k \leq \beta(y) \int_0^{y_0} (y')^k \, \kappa(y', y) \, \mathrm{d}y' \ \leq \ \varepsilon_k \ , \quad \text{a.e. } y \in Y \ ,$$

for $k = 0, 1$, assuming at least ε_1 to be finite. In the following we suppose that

$$\underline{\mu} \ := \ \underset{y \in Y}{\text{ess-inf}} \, \mu(y) \ > \ 0 \tag{64}$$

and

$$\frac{1}{2d_0} \, (\underline{\mu} \, + \, 2\,\delta_0) \ > \ \frac{\tau\lambda}{2\gamma} \, + \, \varepsilon_1 \, - \, 2\,\delta_1 \, + \, \frac{2d_0\delta_1(\varepsilon_1 - \delta_1)}{\underline{\mu} + 2\delta_0} \ . \tag{65}$$

Given the assumptions above we can construct a Lyapunov function as follows.

Lemma 5.1. *Suppose* (62) *or* (63) *and that* (64) *and* (65) *are satisfied. Then there are constants $a, b, p, q > 0$ such that for*

$$F(v, u) := \left(v - \frac{\lambda}{\gamma} \right)^2 \, + \, a \int_{y_0}^\infty y \, u(y) \, \mathrm{d}y \, + \, b \int_{y_0}^\infty u(y) \, \mathrm{d}y$$

there holds

$$F(v, u)(t) \, + \, p \int_0^t \int_{y_0}^\infty u(s, y) \, \mathrm{d}y \, \mathrm{d}s \, + \, q \int_0^t \int_{y_0}^\infty y \, u(s, y) \, \mathrm{d}y \, \mathrm{d}s \ \leq \ F(v^0, u^0)$$

for each $t \geq 0$, where (v, u) is either the classical solution or the weak solution constructed in Theorem 3.1 or Theorem 4.3, respectively.

Proof. Defining

$$A := \frac{\tau}{2}\,(\underline{\mu} + 2\,\delta_0) > 0\ , \quad B := 2\,\delta_1 - \varepsilon_1 - \frac{\tau\,\lambda}{2\,\gamma}\ , \quad C := 4\,\delta_1\,(\varepsilon_1 - \delta_1) \geq 0$$

and $d := \tau d_0/4$, (65) can be recast as

$$\frac{A}{4d} > -B + \frac{Cd}{A}\ .$$

Therefore, with

$$b := \frac{A}{4d^2} + \frac{C}{A} > \frac{C}{A} \geq 0$$

we have $bd < B + \sqrt{Ab - C}$, hence

$$0 < \frac{4}{\tau}\,b\,d < a < \frac{4}{\tau}\big(B + \sqrt{Ab - C}\big) \quad \text{and} \quad \frac{4}{\tau}\big(B - \sqrt{Ab - C}\big) < a \tag{66}$$

for $a := 2/\tau\big(\max\{bd, B - \sqrt{Ab - C}\} + B + \sqrt{Ab - C}\big)$. We set

$$R := b\,(\underline{\mu} + 2\,\delta_0) + \frac{4\,\lambda\,\delta_1}{\gamma} - \frac{\tau\,\lambda^2}{2\,\gamma^2} - \frac{2\,\varepsilon_1^2}{\tau} - \frac{2\,\lambda\,\varepsilon_1}{\gamma}$$

and notice that $0 < Ab - C = B^2 + \tau R/2$, hence $p := -\tau a^2/8 + Ba + R > 0$ by (66). Since (66) also warrants that $d_0 < a/b$, we infer from (64) the existence of $q > 0$ such that

$$\operatorname*{ess\text{-}sup}_{y\in Y} \frac{\beta(y)}{y\mu(y)} + \frac{q}{b}\operatorname*{ess\text{-}sup}_{y\in Y} \frac{1}{\mu(y)} < \frac{a}{b}\ . \tag{67}$$

Now, in the case of the classical solution one can show directly that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(v,u)(t) \leq -p\,|u(t)|_1 - q\int_{y_0}^{\infty} u(t,y)\,y\,\mathrm{d}y\ , \quad t \geq 0\ ,$$

using estimates very close to the subsequent ones. We hence focus on the case of weak solutions. Let (v_n, u_n) be the approximations of (v, u) corresponding to the data $(v^0, u_n^0, \beta_n, \mu_n)$ as in the proof of Theorem 4.3. Then it follows from (12), (14) and (18) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F(v_n, u_n) = &-2\,\gamma\left(v_n - \frac{\lambda}{\gamma}\right)^2 - 2\,\tau\,v_n^2\,|u_n|_1 + \frac{2\tau\lambda}{\gamma}\,v_n\,|u_n|_1\\
&+ 4\left(v_n - \frac{\lambda}{\gamma}\right)\int_{y_0}^{\infty} u_n(y)\,\beta_n(y)\int_0^{y_0} y'\,\kappa(y', y)\,\mathrm{d}y'\,\mathrm{d}y\\
&+ a\,\tau\,v_n\,|u_n|_1 - a\int_{y_0}^{\infty} y\,\mu_n(y)\,u_n(y)\,\mathrm{d}y\\
&- 2\,a\int_{y_0}^{\infty} u_n(y)\,\beta_n(y)\int_0^{y_0} y'\,\kappa(y', y)\,\mathrm{d}y'\,\mathrm{d}y\\
&- b\int_{y_0}^{\infty} \mu_n(y)\,u_n(y)\,\mathrm{d}y\\
&+ b\int_{y_0}^{\infty} u_n(y)\,\beta_n(y)\left(1 - 2\int_0^{y_0} \kappa(y', y)\,\mathrm{d}y'\right)\mathrm{d}y\ .
\end{aligned}$$

Recalling that $\underline{\mu} > 0$ and $\varepsilon_1 < \infty$, integration of the above equality yields (for $n > \underline{\mu}$)

$$\begin{aligned} F(v_n, u_n)(t) &+ \int_0^t |u_n(s)|_1 \left(2\,\tau\, v_n(s)^2 + b\,\underline{\mu}\right) \mathrm{d}s \\ &+ \int_0^t \int_{y_0}^\infty u_n(s,y)\,\beta_n(y) \left[\left(\frac{4\lambda}{\gamma} + 2\,a\right) \int_0^{y_0} y'\,\kappa(y',y)\,\mathrm{d}y' \right. \\ &\qquad \left. + 2\,b \int_0^{y_0} \kappa(y',y)\,\mathrm{d}y' \right] \mathrm{d}y\,\mathrm{d}s \\ &+ a \int_0^t \int_{y_0}^\infty y\,\mu_n(y)\,u_n(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ &\le F(v^0, u_n^0) + b \int_0^t \int_{y_0}^\infty u_n(s,y)\,\beta_n(y)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad + \int_0^t |u_n(s)|_1\, v_n(s)\,\mathrm{d}s \left(\frac{2\tau\lambda}{\gamma} + a\,\tau + 4\,\varepsilon_1\right) . \end{aligned} \tag{68}$$

Observe then that (57) ensures

$$F(v,u)(t) \le \varlimsup_{n\to\infty} F(v_n,u_n)(t) , \quad t \ge 0 . \tag{69}$$

Next, (57) and Lebesgue's theorem imply

$$\lim_{n\to\infty} \int_0^t |u_n(s)|_1\, v_n(s)\,\mathrm{d}s = \int_0^t |u(s)|_1\, v(s)\,\mathrm{d}s , \quad t \ge 0 . \tag{70}$$

As in (61) one shows that

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \int_{y_0}^\infty u_n(s,y)\,\beta_n(y) \int_0^{y_0} (y')^k\,\kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \\ = \int_0^t \int_{y_0}^\infty u(s,y)\,\beta(y) \int_0^{y_0} (y')^k\,\kappa(y',y)\,\mathrm{d}y'\,\mathrm{d}y\,\mathrm{d}s \end{aligned} \tag{71}$$

for $k = 0, 1$. Owing to Lemma 4.2, (48), (52) and (57) we may apply Lebesgue's theorem to conclude

$$\begin{aligned} \lim_{n\to\infty} \int_0^t \int_{y_0}^\infty \chi_{(y_0,R)}(y)\,u_n(s,y)\,\mu_n(y)\,y\,\mathrm{d}y\,\mathrm{d}s \\ = \int_0^t \int_{y_0}^\infty \chi_{(y_0,R)}(y)\,u(s,y)\,\mu(y)\,y\,\mathrm{d}y\,\mathrm{d}s \end{aligned}$$

for each $R > y_0$, hence

$$\int_0^t \int_{y_0}^\infty u(s,y)\,\mu(y)\,y\,\mathrm{d}y\,\mathrm{d}s \le \varlimsup_{n\to\infty} \int_0^t \int_{y_0}^\infty u_n(s,y)\,\mu_n(y)\,y\,\mathrm{d}y\,\mathrm{d}s . \tag{72}$$

Thus, in view of (69)-(72) we may pass to the limit in (68) to deduce that this inequality is still true if we replace (v_n, u_n) by (v, u) and (β_n, μ_n) by (β, μ), respectively. Rearranging the terms and using the definition of δ_k we derive

$$\begin{aligned} F(v,u)(t) \,+ \int_0^t |u(s)|_1 \Big\{ 2\,\tau\, v(s)^2 \,-\, \Big(\frac{2\tau\lambda}{\gamma} + 4\,\varepsilon_1 \,+\, a\,\tau\Big)\, v(s) \\ +\, b\,(\underline{\mu} + 2\delta_0) \,+\, \Big(\frac{4\lambda}{\gamma} \,+\, 2\,a\Big)\,\delta_1 \Big\}\,\mathrm{d}s \\ +\int_0^t \int_{y_0}^\infty \big(a\,y\,\mu(y) \,-\, b\,\beta(y)\big)\, u(s,y)\,\mathrm{d}y\,\mathrm{d}s \\ \le\, F(v^0, u^0) \end{aligned}$$

for each $t \ge 0$. Minimizing the quadratic function in the curly brackets and observing then that $p > 0$ is a lower bound, the assertion follows from (67). □

Remark 5.2. *In the case of rates subject to* (5) *it has already been observed in* [4] *that the function F defined in Lemma 5.1 is a Lyapunov function.*

The next theorem shows that the disease-free steady state is asymptotically stable.

Theorem 5.3. *Suppose* (62) *or* (63) *is satisfied and that* (64), (65) *hold. Then, given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|v(t) - \lambda/\gamma| \,+\, \|u(t)\|_{E_0} \,\le\, \varepsilon\ , \quad t \ge 0\ ,$$

whenever

$$|v^0 - \lambda/\gamma| \,+\, \|u^0\|_{E_0} \,\le\, \delta\ ,$$

where (v, u) is either the classical solution or the weak solution constructed in Theorem 3.1 or Theorem 4.3, respectively.
Moreover, if $\beta(y) \le By$ for a.e. $y \in Y$ and some $B > 0$, then

$$\big(v(t), u(t)\big) \longrightarrow (\lambda/\gamma, 0) \quad \text{in} \quad \mathbb{R} \times L_1(Y, y^\sigma \mathrm{d}y) \quad \text{as} \quad t \longrightarrow \infty$$

for each $\sigma < 1$ and any initial value (v^0, u^0) subject to (62) *or* (63).

Proof. Defining F as in Lemma 5.1, the first statement readily follows from the fact that $F(v,u)(t) \le F(v^0, u^0)$ for $t \ge 0$. Next, Lemma 5.1 also ensures that

$$\|u(t)\|_{L_1(Y, y\mathrm{d}y)} \,\le\, \frac{1}{a}\, F(v^0, u^0)\ , \quad t \ge 0\ . \tag{73}$$

Furthermore, by definition of a weak solution we have

$$|u(t)|_1 \,=\, |u^0|_1 \,+ \int_0^t \int_{y_0}^\infty L[u(s)](y)\,\mathrm{d}y\,\mathrm{d}s\ , \quad t \ge 0\ ,$$

from which we infer that

$$\begin{aligned} \frac{1}{h}\big(|u(t+h)|_1 \,-\, |u(t)|_1\big) \,&=\, \frac{1}{h}\int_t^{t+h} \int_{y_0}^\infty L[u(s)](y)\,\mathrm{d}y\,\mathrm{d}s \\ &\le\, \frac{1}{h}\int_t^{t+h} \int_{y_0}^\infty u(s,y)\,\beta(y)\,\mathrm{d}y\,\mathrm{d}s \\ &\le\, B\,\sup_{s \ge 0} \|u(s)\|_{L_1(Y, y\mathrm{d}y)}\ , \end{aligned}$$

for $t \geq 0$ and $h > 0$. Thus, (73) yields

$$|u(t+h)|_1 - |u(t)|_1 \leq c\,h\ , \quad t\,,\,h > 0\ . \tag{74}$$

Lemma 5.1 also ensures that

$$\int_0^\infty |u(s)|_1\,\mathrm{d}s \leq \frac{1}{p}\,F(v^0,u^0)\ . \tag{75}$$

Combining (74) and (75) we conclude that $\lim_{t\to\infty} |u(t)|_1 = 0$, which, together with (73), warrants that for each $\sigma < 1$

$$u(t) \longrightarrow 0 \quad \text{in} \quad L_1(Y, y^\sigma \mathrm{d}y) \quad \text{as} \quad t \longrightarrow \infty\ . \tag{76}$$

Finally, since $\varepsilon_1 < \infty$ both $g(u(t))$ and $|u(t)|_1$ tend to 0 as $t \to \infty$ due to (76). Since $v \in C^1(\mathbb{R}^+)$ solves (1), it is easy to check that $v(t)$ converges to λ/γ. □

The result above can be improved in the case of classical solutions as follows.

Corollary 5.4. *Suppose* (62)*,* (64)*, and* (65) *hold. Then the classical solution* (v,u) *corresponding to* $v^0 > 0$ *and* $u^0 \in E_1^+$ *satisfies*

$$(v,u) \longrightarrow (\lambda/\gamma, 0) \quad \textit{in} \quad \mathbb{R} \times L_1(Y, y\mathrm{d}y) \quad \textit{as} \quad t \longrightarrow \infty\ .$$

Proof. Set

$$Q(t) := \int_{y_0}^\infty y\,u(t,y)\,\mathrm{d}y \geq 0\ , \quad t \geq 0\ .$$

Then $Q \in C^1(\mathbb{R}^+)$ according to Theorem 3.1. From Lemma 5.1 it follows that

$$Q(t) + \int_0^\infty Q(s)\,\mathrm{d}s \leq c\ , \quad t \geq 0\ . \tag{77}$$

In addition, $v(t) \leq c$ for each $t \geq 0$, whence

$$\dot{Q}(t) \leq \tau\,v(t)\,|u(t)|_1 \leq c\ , \quad t \geq 0\ . \tag{78}$$

Consequently, we deduce $\lim_{t\to\infty} Q(t) = 0$ from (77) and (78). □

Remarks 5.5. *(a) As was pointed out in the introduction, equations* (1)*,* (2) *are no longer coupled in case the rates are subject to* (5)*, since* v *is then completely determined for all* $t \geq 0$*. In this case the results in* [3] *yield a semiflow in the natural phase space* $\mathbb{R}^+ \times L_1^+(Y, y\mathrm{d}y)$*, whereas Theorem 4.3 guarantees existence of weak solutions only for initial values* $(v^0,u^0) \in \mathbb{R}^+ \times L_1^+(Y, y^\alpha \mathrm{d}y)$ *with* $\alpha > 1$*.*

However, in this particular case it can be easily verified that the function (v,u) *in* (57) *satisfies Definition 4.1 for any initial value* $(v^0,u^0) \in \mathbb{R}^+ \times L_1^+(Y, y\mathrm{d}y)$*, provided one takes test functions* $\varphi \in W_\infty^1(Y)$ *with compact support. For this one should note that* $\lim_{y\to\infty} \varrho(y) = 0$ *is merely required for step (v) in the proof of Theorem 4.3.*

(b) If the kernels are of the form (5)*, then we may take* $d_0 = \beta/\mu$*,* $\delta_0 := \beta y_0$ *and* $\varepsilon_1 := \delta_1 := \beta y_0^2/2$*, so* (65) *is equivalent to* (9)*. We should like to point out that in this case the authors in* [3] *prove that the disease-free steady state is globally exponentially stable in* $\mathbb{R}^+ \times L_1^+(Y, y\mathrm{d}y)$*, and asymptotically stable if* $\beta y_0 + \mu = \sqrt{\beta\lambda\tau/\gamma}$*.*

(c) If the rates are subject to (5) *it has already been observed in* [4] *that system* (1)*–*(2) *admits also a non-trivial (disease) steady state, provided the inequality in*

(9) *is reversed. It is shown in* [3] *that this steady state is again globally asymptotically stable in* $\mathbb{R}^+ \times L_1^+(Y, y\mathrm{d}y)$. *For general rates as in the present publication, existence of other equilibria besides* $(\lambda/\gamma, 0)$ *is an open problem.*

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References

[1] H. Amann. *Linear and quasilinear parabolic problems, volume I: Abstract linear theory.* Birkhäuser, Basel, Boston, Berlin 1995.
[2] R.E. Edwards. *Functional analysis. Theory and applications.* Dover Publ., New York 1995.
[3] H. Engler, J. Prüss, G. Webb. *Analysis of a model for the dynamics of prions II.* J. Math. Anal. Appl. To appear.
[4] M. Greer. *A population model of prion dynamics.* Ph.D. Thesis, Vanderbilt University, Nashville 2002.
[5] M. Greer, L. Pujo-Menjouet, G. Webb. *A mathematical analysis of the dynamics of prion proliferation.* Preprint.
[6] M. Lachowicz, P. Laurençot, D. Wrzosek. *On the Oort-Hulst-Safronov coagulation equation and its relation to the Smoluchowski equation.* SIAM J. Math. Anal. **34**, No. 6 (2003), 1399-1421.
[7] P. Laurençot, S. Mischler. *The continuous coagulation-fragmentation equations with diffusion.* Arch. Rat. Mech. Anal. **162**, No.1 (2002), 45-99.
[8] P. Laurençot, C. Walker. In preparation.
[9] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer, Berlin, New York, Heidelberg 1983.
[10] J. Prüss, L. Pujo-Menjouet, G. Webb, R. Zacher. *Analysis of a model for the dynamics of prions.* Discrete Contin. Dyn. Syst. Ser. B **6** (2006), no. 1, 225–235 (electronic).
[11] I.W. Stewart. *A global existence theorem for the general coagulation-fragmentation equation with unbounded kernels.* Math. Meth. Appl. Sci. **11** (1989), 627-648.
[12] I.I. Vrabie. *Compactness methods for nonlinear evolutions.* 2^{nd} edition. Longman, London 1995.
[13] C. Walker. *On diffusive and non-diffusive coalescence and breakage processes.* Doctoral Thesis, Universität Zürich, 2003.

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