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Coalescence and breakage processes

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SUMMARY

We extend a model for coalescence and breakage of liquid–liquid dispersions proposed by Fasano/Rosso. The main feature is that the experimentally observed fact of a maximal droplet mass is taken into account. Our model includes spontaneous breakage as well as collisional fragmentation. Existence and uniqueness of solutions is proved and the long-time behaviour is investigated. Copyright © 2002 John Wiley & Sons, Ltd.

1. INTRODUCTION

During recent years much effort has been invested in the study of coagulation and fragmentation processes which arise in a variety of situations such as colloid chemistry, polymer and aerosol science. In those models, a system of a large number of clusters is considered where the clusters can merge to form larger particles or break into smaller ones.

In the present paper our attention is focused on an extension of a new model, introduced for the first time by Fasano and Rosso [1], describing the evolution of droplets of a liquid–liquid dispersion in a batch reactor. This model involves new features taking into account some experimentally observed facts. Most important is the existence of a maximal droplet mass or volume (cf. References [2–4]) which requires a new interaction mechanism, called *volume scattering*, to prevent the occurrence of droplets that are 'too large'. The underlying idea is that, if two droplets with cumulative mass exceeding the maximal droplet mass collide, the formed cluster immediately decays into droplets all with mass within the admissible range. As we shall see, this assumption complicates the statement of the problem and questions concerning long-time behaviour while it avoids some mathematical difficulties such as summability. Another new feature is the introduction of an efficiency factor linked to some average properties of the dispersion. Furthermore, we consider two different types of fragmentation. The most common one, known as *spontaneous breakage*, results from external forces such as turbulent pressure fluctuations in the vicinity of a droplet or shear forces. In the other case, if two droplets collide they either coalesce or a shattering of the droplets occurs—which is called

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collisional breakage—and has hardly been investigated mathematically so far (however, see References [5, 6]).

To be more precise, let $u(t,x)$ be the distribution function of droplet size at time t (per unit mass), where x denotes a characteristic of the droplet such as mass or volume. We assume droplets to be uniformly distributed so that u is independent of spatial co-ordinates. This seems to be reasonable in a batch reactor with sufficiently high shear rate. By $x_0 \in (0,\infty)$ we denote the maximal droplet mass (or volume) which depends on several parameters (shear rate, reactor geometry, temperature, and others). Then the evolution of the system of droplets that undergo both coalescence and breakage can be described by the integro-differential equation

$$\begin{aligned} \dot{u}(x) &= \varphi(u)\{L_\mathrm{b}(u)(x) + L_\mathrm{c}(u)(x) + L_\mathrm{s}(u)(x)\}, \quad t>0 \\ u(0,x) &= u^0(x) \end{aligned} \tag{$*$}$$

for $x \in (0,x_0]$, where u^0 is a given initial distribution, and the operators in $(*)$ are defined as

$$L_\mathrm{b}(u)(x) := \int_x^{x_0} \gamma(y,x)u(y)\,\mathrm{d}y - u(x)\int_0^x \frac{y}{x}\gamma(x,y)\,\mathrm{d}y$$

$$\begin{aligned} L_\mathrm{c}(u)(x) := {} & \frac{1}{2}\int_0^x K(y,x-y)P(y,x-y)u(y)u(x-y)\,\mathrm{d}y \\ & + \frac{1}{2}\int_x^{x_0}\int_0^y K(z,y-z)(1-P(z,y-z))\beta(y,x)u(z)u(y-z)\,\mathrm{d}z\,\mathrm{d}y \\ & - u(x)\int_0^{x_0-x} K(x,y)u(y)\,\mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} L_\mathrm{s}(u)(x) := {} & \frac{1}{2}\int_{x_0}^{2x_0}\int_{y-x_0}^{x_0} K(z,y-z)\beta(y,x)u(z)u(y-z)\,\mathrm{d}z\,\mathrm{d}y \\ & - u(x)\int_{x_0-x}^{x_0} K(x,y)u(y)\,\mathrm{d}y \end{aligned}$$

The operator $L_\mathrm{b}(u)$ gives the gain and loss of droplets of mass x due to multiple spontaneous breakage where the kernel $\gamma(x,y)$ represents the rate at which a droplet of mass x decays in a droplet of mass $y \in (0,x)$.

The collision operator $L_\mathrm{c}(u)$ reflects the possible events that happen if two droplets x and y with cumulative mass $x+y \leqslant x_0$ collide. They either coalesce with probability $P(x,y)$ or the virtual droplet $x+y$ breaks into several fragments with probability $1-P(x,y)$. The symmetric function $K(x,y)$ denotes the rate of binary collision, and $\beta(x+y,z)$ is the distribution of products from a particle $x+y$ breaking after collision. Here, β depends only on the cumulative mass $x+y$ although it would make only a slight difference in the further analysis to allow β to depend on each colliding droplet x and y. The factors $\frac{1}{2}$ are due to symmetry.

Lastly, the ‘scattering’ operator $L_s(u)$ describes the interaction of pairs of droplets whose cumulative mass exceeds x_0 and decay immediately in droplets with mass within the admissible range $(0, x_0]$. This action corresponds to $P \equiv 0$ in the collisional breakage process.

The idea of the factor $\varphi(u)$ is to enhance or depress the dynamics while the mechanical structure of the interactions is described by the kernels K, γ and β. For instance, $\varphi(u)$ may be of the form

$$\varphi(u) = \Phi\left(\int_0^{x_0} u(x)\,\mathrm{d}x, \int_0^{x_0} x^{2/3} u(x)\,\mathrm{d}x\right)$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is a given function. This means that $\varphi(u)$ is related to the total number of droplets and the total surface area, respectively.

As mentioned above this model is adapted from those of Fasano and Rosso [1] but includes some extensions. In their model they consider the case of pure binary spontaneous breakage only, i.e. $P \equiv 1$ in $(*)$, and each droplet decays (if it does) just in two fragments. But if binary breakage is considered and $P \equiv 1$ then it is reasonable to assume that

$$\gamma(x, y) = \gamma(x, x - y), \quad 0 < y < x \leqslant x_0 \tag{1}$$

$$\beta(x, y) = \beta(x, x - y), \quad x_0 < x \leqslant 2x_0, \quad x - x_0 < y \leqslant x_0 \tag{2}$$

and

$$\beta(x, y) = 0, \quad 0 < y < x - x_0 \tag{3}$$

Indeed, if a droplet of mass x decays in a fragment y then also a droplet of mass $x - y$ is formed. On the other hand, each one of the fragments y and $x - y$ created by the decay of $x \in (x_0, 2x_0]$ has to belong to $(0, x_0]$. Therefore, (3) is due to consistency of our model. By a change of variable we see that (1) leads to

$$\int_0^x \frac{y}{x}\gamma(x, y)\,\mathrm{d}y = \frac{1}{2}\int_0^x \gamma(x, y)\,\mathrm{d}y, \quad 0 < x \leqslant x_0 \tag{4}$$

if both integrals exist. Similarly, provided breakage conserves the mass (see Reference [1] and also our Hypothesis (H_3) below), i.e.,

$$\int_0^{x_0} y\beta(x, y)\,\mathrm{d}y = x, \quad x_0 < x \leqslant 2x_0$$

we have due to (2) and (3)

$$1 = \int_0^{x_0} \frac{y}{x}\beta(x, y)\,\mathrm{d}y = \frac{1}{2}\int_{x-x_0}^{x_0} \beta(x, y)\,\mathrm{d}y, \quad x_0 < x \leqslant 2x_0 \tag{5}$$

Consequently, under the assumption of pure binary spontaneous breakage, which means that $P \equiv 1$ and that (1)–(3) are satisfied, the system considered in Reference [1] is equivalent to $(*)$.

To conclude the presentation of our model let us remark the following: although it may be unreasonable from a physical point of view to assume that spontaneous and collisional breakage occur at the same time, we decided to treat them simultaneously since most of the results obtained exclude neither the case $P \equiv 1$ (pure spontaneous breakage) nor $\gamma \equiv 0$ (pure collisional breakage).

In our approach, we interpret $(*)$ as an ordinary differential equation in the Banach space $L_1 := L_1((0,x_0])$, i.e.,

$$\begin{aligned} \dot{u} &= \varphi(u)\{L_{\mathrm{b}}[u] + L_{\mathrm{c}}[u,u] + L_{\mathrm{s}}[u,u]\} \quad \text{in } L_1, \quad t>0 \\ u(0) &= u^0 \end{aligned} \tag{$**$}$$

where we introduced for $u, v \in L_1$, the notations

$$\begin{aligned} L_{\mathrm{b}}[u](x) &:= L_{\mathrm{b}}^1[u](x) - L_{\mathrm{b}}^2[u](x) \\ &:= \int_x^{x_0} \gamma(y,x)u(y)\,\mathrm{d}y - u(x)\int_0^x \frac{y}{x}\gamma(x,y)\,\mathrm{d}y \\ L_{\mathrm{c}}[u,v](x) &:= L_{\mathrm{c}}^1[u,v](x) + L_{\mathrm{c}}^2[u,v](x) - L_{\mathrm{c}}^3[u,v](x) \\ &:= \frac{1}{2}\int_0^x K(y,x-y)P(y,x-y)u(y)v(x-y)\,\mathrm{d}y \\ &\quad + \frac{1}{2}\int_x^{x_0}\int_0^y K(z,y-z)(1-P(z,y-z))\beta(y,x)u(z)v(y-z)\,\mathrm{d}z\,\mathrm{d}y \\ &\quad - u(x)\int_0^{x_0-x} K(x,y)v(y)\,\mathrm{d}y \\ L_{\mathrm{s}}[u,v](x) &:= L_{\mathrm{s}}^1[u,v](x) - L_{\mathrm{s}}^2[u,v](x) \\ &:= \frac{1}{2}\int_{x_0}^{2x_0}\int_{y-x_0}^{x_0} K(z,y-z)\beta(y,x)u(z)v(y-z)\,\mathrm{d}z\,\mathrm{d}y \\ &\quad - u(x)\int_{x_0-x}^{x_0} K(x,y)v(y)\,\mathrm{d}y \end{aligned}$$

In the sequel, we put $\|\cdot\| := \|\cdot\|_{L_1}$ and assume throughout this paper that the following hypotheses are satisfied:

(H_1) $\varphi : L_1 \to \mathbb{R}^+$ is bounded and Lipschitz continuous on bounded sets,

(H_2) γ is a measurable function from $\Delta := \{(x,y); 0<y<x\leqslant x_0\}$ into $\mathbb{R}^+$ and there exists $m_\gamma > 0$ with

$$\int_0^x \gamma(x,y)\,\mathrm{d}y \leqslant m_\gamma \quad \text{for a.a. } 0<x\leqslant x_0$$

(H_3) β is a measurable function defined on

$$\Lambda := \{(x,y); 0<y<x\leqslant x_0\} \cup \{(x,y); 0<y\leqslant x_0 < x \leqslant 2x_0\}$$

with values in $\mathbb{R}^+$, and there exists $m_\beta \geqslant 2$ with

$$\int_0^{\min\{x,x_0\}} \beta(x,y)\,\mathrm{d}y \leqslant m_\beta \quad \text{for a.a. } 0<x\leqslant 2x_0$$

and

$$\int_0^{\min\{x,x_0\}} y\beta(x,y)\,\mathrm{d}y = x \quad \text{for a.a. } 0<x\leqslant 2x_0$$

(H_4) $P, K \in L_\infty((0,x_0]^2, \mathbb{R}^+)$ are symmetric and $0 \leqslant P \leqslant 1$.

Although multiple breakage is allowed in our model, Hypotheses (H_2) and (H_3) imply that only a limited number of daughter droplets are produced by rupture. Since the first integral of (H_3) represents the expected number of droplets resulting from breakage it is *a priori* bounded from below by 2. The second condition of (H_3) means conservation of mass. Note that these assumptions are weaker than those of Reference [1] (if we put $P\equiv 1$, of course).

2. EXISTENCE

In this section we prove existence and uniqueness of a maximal non-negative solution which preserves the total mass. In some cases, e.g. in the case of binary breakage or in the case of pure spontaneous breakage, this solution exists globally.

The following lemma can be proved using Fubini's theorem, a suitable change of variables and Hypotheses (H_2)–(H_4).

Lemma 2.1
The operator $L_\mathrm{b}[\cdot]: L_1 \to L_1$ is linear and $L_h[\cdot,\cdot]: L_1 \times L_1 \to L_1$ is bilinear for $h = c, s$. Moreover, for $u, v \in L_1$ the following estimates hold:

(i) $\|L_\mathrm{b}[u]\| \leqslant 2m_\gamma \|u\|$
(ii) $\|L_\mathrm{c}[u,v]\| \leqslant \frac{1}{2}(m_\beta + 3)\|K\|_\infty \|u\|\,\|v\|$
(iii) $\|L_\mathrm{s}[u,v]\| \leqslant \frac{1}{2}(m_\beta + 2)\|K\|_\infty \|u\|\,\|v\|$

Theorem 2.2
For each $u^0 \in L_1$ there exists a unique maximal solution

$$u := u(\cdot\,; u^0) \in C^1(J(u^0), L_1)$$

for $(**)$, where the maximal existence interval $J(u^0)$ is open in $\mathbb{R}^+$. If $t^+(u^0) := \sup J(u^0) < \infty$ then

$$\lim_{t \nearrow t^+(u^0)} \|u(t;u^0)\| = \infty \tag{6}$$

Moreover, the map $[(t,u^0) \mapsto u(t;u^0)]$ generates a semiflow on L_1.

Proof
Since $L_\mathrm{b}[\cdot]$ is linear and $L_h[\cdot,\cdot]$ is bilinear for $h \in \{\mathrm{c},\mathrm{s}\}$, Hypothesis ($H_1$) and Lemma 2.1 imply that the map

$$L_1 \to L_1, \quad u \mapsto \varphi(u)\{L_\mathrm{b}[u] + L_\mathrm{c}[u,u] + L_\mathrm{s}[u,u]\}$$

is Lipschitz continuous and bounded on bounded sets. Thus, standard arguments from the theory of ordinary differential equations (see Reference [7]) lead to the assertion. □

Our proof of existence of a solution, which is continuously differentiable in t and belongs to L_1 at any time, is based on a fixed point argument and differs from that given by Fasano and Rosso [1]. By using the Arzelà–Ascoli Theorem, they show more regularity with respect to the droplet size, i.e., they prove that the solution is Lipschitz continuous in $x\in(0,x_0]$. But, to do so, more restrictive conditions are required. In particular, they impose that the kernels and the initial value are (piecewise) continuously differentiable.

Corollary 2.3
Assume either $\varphi\equiv\text{const}$ or $\varphi: L_1\to\mathbb{R}^+$ is continuous and linear. Then, for any $u^0\in L_1$

$$u=u(\cdot;u^0)\in C^\infty(J(u^0),L_1)$$

Proof
Since $u\in C^1(J(u^0),L_1)$ and $L_\mathrm{b}[\cdot]$ is linear and continuous, $L_\mathrm{b}[u]$ belongs to $C^1(J(u^0),L_1)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}L_\mathrm{b}[u(t)]=L_\mathrm{b}[\dot u(t)],\quad t\in J(u^0)$$

and similarly $L_h[u,u]\in C^1(J(u^0),L_1)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}L_h[u(t),u(t)]=L_h[\dot u(t),u(t)]+L_h[u(t),\dot u(t)],\quad t\in J(u^0),\quad h=\mathrm{c},\mathrm{s}$$

The right-hand side of $(**)$ is therefore continuously differentiable and we conclude that $u\in C^2(J(u^0),L_1)$. The assertion follows now by induction. □

In the sequel, for $u^0\in L_1$ given, we denote by $u\in C^1(J(u^0),L_1)$ the unique maximal solution for $(**)$ and put

$$\varphi(t):=\varphi(u(t)),\quad t\in J(u^0)$$

Furthermore, L_1^+ is the closed subset of L_1 consisting of all $v\in L_1$ which are non-negative almost everywhere.

Theorem 2.4
For any initial distribution $u^0\in L_1^+$, the solution $u(t;u^0)$ remains non-negative, i.e.,

$$u(t;u^0)\in L_1^+,\quad t\in J(u^0)$$

Proof
We choose any $T_0\in J(u^0)\backslash\{0\}$ and put

$$\|\varphi\|_\infty:=\max_{0\leqslant t\leqslant T_0}|\varphi(t)|$$

$$\|u\|_\infty:=\max_{0\leqslant t\leqslant T_0}\|u(t)\|$$

as well as

$$\omega:=\|\varphi\|_\infty(m_\gamma+2\|K\|_\infty\|u\|_\infty)\geqslant 0$$

For $0 \leqslant t \leqslant T \leqslant T_0$ and $v \in C([0,T],L_1)$ set

$$G(t,v(t)) := \varphi(t)\{L_{\mathrm{b}}[v(t)] + L_{\mathrm{c}}^1[v(t),v(t)] + L_{\mathrm{c}}^2[v(t),v(t)] + L_{\mathrm{s}}^1[v(t),v(t)] - L_{\mathrm{c}}^3[v(t),u(t)] - L_{\mathrm{s}}^2[v(t),u(t)]\} + \omega v(t)$$

Then $G(\cdot,v(\cdot)) \in C([0,T],L_1)$ due to Lemma 2.1. Further, there exists $c(T_0) > 0$ with

$$\|G(t,v(t)) - G(t,w(t))\| \leqslant c(T_0)(1 + \|v(t)\| + \|w(t)\|)\|v(t) - w(t)\| \tag{7}$$

for $v,w \in C([0,T],L_1)$ and $0 \leqslant t \leqslant T$. Since $v(t) \in L_1^+$ implies

$$L_{\mathrm{b}}^1[v(t)],\ L_{\mathrm{c}}^2[v(t),v(t)],\ L_h^1[v(t),v(t)] \in L_1^+, \quad h = \mathrm{c},\mathrm{s}$$

it follows that

$$G(t,v(t)) \geqslant [-\|\varphi\|_\infty(m_\gamma + 2\|K\|_\infty\|u\|_\infty) + \omega]v(t) = 0 \quad \text{a.e.} \tag{8}$$

for any such v. Now put $p := \|u\|_\infty + 2$ and choose $T \in (0,T_0]$ such that

$$c(T_0)(p + p^2)T < 1$$

Due to (7), the definition of

$$F(v)(t) := \mathrm{e}^{-\omega t}u^0 + \int_0^t \mathrm{e}^{-\omega(t-s)}G(s,v(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T$$

for

$$v \in \mathscr{V}_T := \{v \in C([0,T],L_1);\, \|v(t)\| \leqslant p,\ 0 \leqslant t \leqslant T\}$$

yields a contraction $F : \mathscr{V}_T \to \mathscr{V}_T$ with constant

$$c(T_0)(1 + 2p)T < c(T_0)(p + p^2)T < 1$$

On the other hand, u, being a solution of $(**)$, solves

$$\begin{aligned} \dot{v} + \omega v &= G(t,v), \quad 0 < t \leqslant T \\ v(0) &= u^0 \end{aligned}$$

as well and belongs to $\mathscr{V}_T$. Thus, u is the unique fixed point of F. Putting

$$u_0 := u^0 \in \mathscr{V}_T, \qquad u_{n+1} := F(u_n) \in \mathscr{V}_T, \quad n \in \mathbb{N}$$

we have by induction and (8) that $u_n(t) \geqslant 0$ a.e. for all $n \in \mathbb{N}$ and $0 \leqslant t \leqslant T$. Since $u_n \to u$ in $\mathscr{V}_T$, this implies

$$u(t) \geqslant 0 \text{ a.e.}, \quad 0 \leqslant t \leqslant T$$

Put $T^* := \sup\{T > 0; u(t) \geqslant 0 \text{ a.e. for } 0 \leqslant t \leqslant T\}$, assume $T^* < T_0$, and consider

$$\begin{aligned} \dot{v} + \omega v &= G(t + T^*, v), \quad 0 < t \leqslant T_0 - T^* \\ v(0) &= u(T^*) \end{aligned} \tag{9}$$

Then $u(T^*) \in L_1^+$ since L_1^+ is closed in L_1 and, further, $u(\cdot + T^*)$ is a solution of (9). By repeating the above arguments we conclude

$$u(t + T^*) \geqslant 0 \text{ a.e.}, \quad 0 \leqslant t \leqslant T$$

for a suitable $T > 0$. But this contradicts our choice of T^*. Thus $T^* = T_0$ and, $T_0 \in J(u^0)\backslash\{0\}$ being arbitrary, the assertion follows. $\square$

Remark 2.5
Theorems 2.2 and 2.4 guarantee that the map

$$[(t, u^0) \mapsto u(t; u^0)]$$

generates a semiflow on L_1^+.

Lemma 2.6
For any $f \in L_\infty((0, x_0])$ and $v \in L_1$ the following identities hold:

(i)

$$\int_0^{x_0} f(x) L_b[v](x)\,dx = \int_0^{x_0} \int_0^x \left\{ f(y) - \frac{y}{x} f(x) \right\} \gamma(x, y)\,dy\ v(x)\,dx$$

(ii)

$$\begin{aligned} \int_0^{x_0} f(x) L_c[v, v](x)\,dx = \frac{1}{2} \int_0^{x_0} \int_0^{x_0 - x} \Big\{ & P(x, y) f(x + y) - f(x) - f(y) \\ & + (1 - P(x, y)) \int_0^{x+y} f(z) \beta(x + y, z)\,dz \Big\} K(x, y) v(y) v(x)\,dy\,dx \end{aligned}$$

(iii)

$$\begin{aligned} & \int_0^{x_0} f(x) L_s[v, v](x)\,dx \\ & \quad = \frac{1}{2} \int_0^{x_0} \int_{x_0 - x}^{x_0} \left\{ \int_0^{x_0} f(z) \beta(x + y, z)\,dz - f(x) - f(y) \right\} K(x, y) v(y) v(x)\,dy\,dx \end{aligned}$$

Proof
The statements are consequences of Fubini's theorem and suitable changes of variables whereby all of the integrals remain finite due to Hypotheses (H_2)–(H_4). For (ii) and (iii) recall that K and P are symmetric. □

Remark 2.7
Suppose $f \equiv 1$. Then, for $v \in L_1^+$, Lemma 2.6 reflects the intuitively evident facts that breakage and scattering increase the total number of droplets. If only binary breakage is considered then scattering does not alter the number of droplets since (H_3) and (2) imply

$$\int_0^{x_0} \beta(x,z)\,\mathrm{d}z = 2, \quad \text{a.a. } x_0 < x \leqslant 2x_0$$

as shown in the introduction. Thus,

$$\int_0^{x_0} L_\mathrm{s}[v,v](x)\,\mathrm{d}x = 0$$

in this case. Whether collision increases or decreases the total number of clusters depends on the probability of coalescence and on the number of fragments formed if two droplets do not coalesce after collision. However, if $P \equiv 1$ or if two colliding droplets break into two fragments only, the total number of droplets is reduced by this mechanism.

As an immediate consequence of the preceding lemma, any solution of ($**$) conserves the total mass.

Theorem 2.8
Let $u^0 \in L_1$. Then, for any $t \in J(u^0)$,

$$\int_0^{x_0} xu(t;u^0)(x)\,\mathrm{d}x = \int_0^{x_0} xu^0(x)\,\mathrm{d}x$$

Proof
For $t \in J(u^0)$ we have

$$u(t) = u^0 + \int_0^t \varphi(s)\{L_\mathrm{b}[u(s)] + L_\mathrm{c}[u(s),u(s)] + L_\mathrm{s}[u(s),u(s)]\}\,\mathrm{d}s$$

Thus [8, p. 69f] gives

$$u(t)(x) = u^0(x) + \int_0^t \varphi(s)\{L_\mathrm{b}[u(s)](x) + L_\mathrm{c}[u(s),u(s)](x) + L_\mathrm{s}[u(s),u(s)](x)\}\,\mathrm{d}s \tag{10}$$

for a.a. $x \in (0,x_0]$. Multiplying both sides with x, integrating then from 0 to x_0 and changing the order of integration, Lemma 2.6 leads to the assertion since each of the processes collision, breakage and scattering is mass preserving in view of Hypothesis (H_3). □

Theorem 2.9
Assume $\|\varphi\|_\infty := \sup_{v \in L_1^+} \varphi(v) < \infty$. Furthermore, let one of the following conditions be satisfied:

(i) $K(x,y) \leqslant K^*(x+y)$ for a.a. $(x,y) \in (0,x_0]^2$ and some $K^* > 0$;

(ii) there exists $z_0 \in (0, x_0]$ such that

$$\int_0^{x+y} \beta(x+y,z)\,\mathrm{d}z \leqslant \frac{2-P(x,y)}{1-P(x,y)} \tag{11}$$

for a.a. $(x,y) \in (0,x_0]^2$ with $x+y \leqslant z_0$.

Then, the solution $u(\cdot;u^0)$ exists globally for $u^0 \in L_1^+$, i.e., $J(u^0) = \mathbb{R}^+$.

Proof

In analogy to the proof of Theorem 2.8 we have

$$\begin{aligned} \|u(t)\| &= \int_0^{x_0} u(t)(x)\,\mathrm{d}x \\ &= \|u^0\| + \int_0^t \varphi(s) \int_0^{x_0} \{L_{\mathrm{b}}[u(s)](x) \\ &\quad + L_{\mathrm{c}}[u(s),u(s)](x) + L_{\mathrm{s}}[u(s),u(s)](x)\}\,\mathrm{d}x\,\mathrm{d}s \end{aligned} \tag{12}$$

for $t \in J(u^0)$ since $u(t)$ is non-negative. Lemma 2.6 leads to the estimate

$$\int_0^t \varphi(s) \int_0^{x_0} L_{\mathrm{b}}[u(s)](x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\varphi\|_\infty m_\gamma \int_0^t \|u(s)\|\,\mathrm{d}s$$

Using

$$K(x,y) \leqslant \frac{\|K\|_\infty}{x_0}(x+y) \quad \text{for a.a. } (x,y) \text{ with } x+y > x_0 \tag{13}$$

and conservation of mass we see that

$$\int_0^t \varphi(s) \int_0^{x_0} L_{\mathrm{s}}[u(s),u(s)](x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\varphi\|_\infty m_\beta \frac{\|K\|_\infty}{x_0} \int_0^{x_0} x u^0(x)\,\mathrm{d}x \int_0^t \|u(s)\|\,\mathrm{d}s$$

If (i) holds then Lemma 2.6 ensures that

$$\int_0^t \varphi(s) \int_0^{x_0} L_{\mathrm{c}}[u(s),u(s)](x)\,\mathrm{d}x\,\mathrm{d}s \leqslant \|\varphi\|_\infty (m_\beta + 2) K^* \int_0^{x_0} x u^0(x)\,\mathrm{d}x \int_0^t \|u(s)\|\,\mathrm{d}s$$

since $0 \leqslant P \leqslant 1$. On the other hand, if (ii) is satisfied then

$$\int_A [P(x,y) - 2 + (1-P(x,y))\nu(x+y)] K(x,y) u(x) u(y)\,\mathrm{d}(x,y) \leqslant 0$$

where we put

$$A := \{(x,y) \in (0,x_0]^2; x+y \leqslant z_0\}$$

and $\nu(x+y) := \int_0^{x+y} \beta(x+y,z)\,\mathrm{d}z$. For

$$B := \{(x,y) \in (0,x_0]^2; z_0 < x+y \leqslant x_0\}$$

we have $K(x,y) \leqslant \|K\|_\infty (x+y)/z_0$ for a.a. $(x,y) \in B$, and as a consequence

$$\begin{aligned} &\int_0^t \varphi(s) \int_0^{x_0} L_c[u,u](x)\,dx\,ds \\ &\quad \leqslant \int_0^t \varphi(s) \int_B |P(x,y) - 2 + (1 - P(x,y))v(x+y)| K(x,y) u(x) u(y)\, d(x,y)\ ds \\ &\quad \leqslant \|\varphi\|_\infty (m_\beta + 2) \frac{\|K\|_\infty}{z_0} \int_0^{x_0} x u^0(x)\,dx \int_0^t \|u(s)\|\,ds \end{aligned}$$

From (12) we thus conclude in both cases that

$$\|u(t)\| \leqslant \|u^0\| + c(u^0) \int_0^t \|u(s)\|\,ds, \quad t \in J(u^0)$$

where $c(u^0) > 0$ does not depend on $t \in J(u^0)$, and therefore, by applying Gronwall's inequality,

$$\|u(t)\| \leqslant \|u^0\| e^{c(u^0)t}, \quad t \in J(u^0)$$

Recalling Theorem 2.2, the proof is complete. $\square$

Remark 2.10
Note that if only spontaneous breakage is considered, i.e. $P \equiv 1$, then the solution $u(\cdot; u^0)$ for $u^0 \in L_1^+$ is global since (11) holds. Because binary breakage and Hypothesis (H_3) imply

$$\int_0^x \beta(x,y)\,dy = 2 \quad \text{for a.a. } 0 < x \leqslant x_0 \tag{14}$$

hence (11), we have global existence in this case also.

3. LONG-TIME BEHAVIOUR

In this section, long-time behaviour is investigated and some *a priori* estimates leading to (in-)stability of the trivial solution to ($**$) are established for certain kernels. It turns out that the evolution of the total number of droplets

$$\int_0^{x_0} u(t; u^0)(x)\,dx$$

is strongly related to the behaviour affected by coalescence and breakage of small droplets. The results obtained reflect expected physical properties.

Throughout this section we assume Hypotheses (H_1)–(H_4) with

$$\|\varphi\|_\infty := \sup_{v \in L_1^+} \varphi(v) < \infty$$

and, for $u^0 \in L_1^+$ given, we denote by

$$u = u(\cdot; u^0) \in C^1(J(u^0), L_1^+)$$

the unique maximal solution for $(**)$. Recall that $u(\cdot;u^0)$ preserves total mass. As before we put

$$\varphi(t) := \varphi(u(t)), \quad t \in J(u^0)$$

To shorten notation, the moments

$$M_\alpha(t) := \int_0^{x_0} x^\alpha u(t)(x)\,\mathrm{d}x, \quad t \in J(u^0)$$

are introduced for $\alpha \geqslant 0$. Then $M_1(t) \equiv M_1(0)$ is equal to the total mass and $M_0(t) = \|u(t)\|$ represents the total number of droplets for $t \in J(u^0)$. Note that, since conservation of mass leads to a lower bound for $M_0(t)$, i.e.

$$M_0(t) \geqslant \frac{M_1(0)}{x_0}, \quad t \in J(u^0)$$

the trivial solution is not attractive for the semiflow $[(t,u^0) \mapsto u(t;u^0)]$ generated on L_1^+.

First, we assume that there exists $x_c \in (0, x_0)$ such that droplets with mass less than x_c are not produced by rupture, i.e.

$$\begin{aligned} \gamma(x,y) &= 0 \quad \text{for a.a. } 0 < y \leqslant x_c \\ \beta(x,y) &= 0 \quad \text{for a.a. } 0 < y \leqslant x_c \end{aligned} \tag{15}$$

If we denote by $v_\gamma(x)$ for $x \in (0, x_0]$ the expected number of fragments resulting from spontaneous breakage of a droplet x and by $v_\beta(x)$ for $x \in (0, 2x_0]$ those resulting from collisional breakage, respectively, then assumption (15) implies

$$\begin{aligned} \gamma(x,y) &= 0 \quad \text{for } x < v_\gamma(x)x_c \\ \beta(x,y) &= 0 \quad \text{for } x < v_\beta(x)x_c \end{aligned}$$

Indeed, if, for example, $x \in (0, x_0]$ would decay into $v_\gamma(x)$ pieces and $x < v_\gamma(x)x_c$ then necessarily at least one of its fragments would have a mass less than x_c.

Consequently, droplets with mass $< x_c$ are those already existing at initial time $t = 0$ and can disappear only by coalescence. This fact implies an upper bound for the total number of droplets and stability of the trivial solution for the semiflow generated on L_1^+.

Theorem 3.1
Assume that (15) holds. Then $u(\cdot;u^0)$ exists globally and

$$\|u(t;u^0)\| \leqslant \left(1 + \frac{x_0}{x_c}\right) \|u^0\|, \quad t \geqslant 0$$

Proof
Condition (15) yields

$$\int_0^{x_c} L_b[u](x)\,\mathrm{d}x = 0$$

and

$$\int_0^{x_c} L_s[u,u](x)\,dx = -\int_0^{x_c}\int_{x_0-x}^{x_0} K(x,y)u(x)u(y)\,dy\,dx \leqslant 0$$

Moreover, since $0 \leqslant P \leqslant 1$ and (15) ensure

$$\begin{aligned}\int_0^{x_c} L_c[u,u](x)\,dx &= \frac{1}{2}\int_0^{x_c}\int_0^{x_c-x} P(x,y)K(x,y)u(x)u(y)\,dy\,dx \\ &\quad -\int_0^{x_c}\int_0^{x_0-x} K(x,y)u(x)u(y)\,dy\,dx \\ &\leqslant -\frac{1}{2}\int_0^{x_c}\int_0^{x_0-x} K(x,y)u(x)u(y)\,dy\,dx \leqslant 0\end{aligned}$$

we conclude from (10) that

$$\int_0^{x_c} u(t;u^0)(x)\,dx \leqslant \int_0^{x_c} u^0(x)\,dx, \quad t \geqslant 0$$

But this entails

$$\begin{aligned}\|u(t)\| &= \int_0^{x_c} u(t;u^0)(x)\,dx + \int_{x_c}^{x_0} u(t;u^0)(x)\,dx \\ &\leqslant \int_0^{x_c} u^0(x)\,dx + \frac{1}{x_c}\int_0^{x_0} xu(t;u^0)(x)\,dx \\ &\leqslant \left(1+\frac{x_0}{x_c}\right)\|u^0\|\end{aligned}$$

for all $t \in J(u^0)$, and consequently $J(u^0) = \mathbb{R}^+$ by Theorem 2.2. □

Lemma 3.2
Let $a,b \geqslant 0$ with $(a,b) \neq (0,0)$, $c>0$ and $f^0>0$ be given.
Put $D := b^2+4ac>0$ and $R := (b+\sqrt{D})/2c$. Then, the unique solution of

$$\begin{aligned}\dot{f} &= a+bf-cf^2, \quad t>0\\ f(0) &= f^0\end{aligned}$$

is given by

$$f(t) = \begin{cases} \frac{b}{2c}+\frac{\sqrt{D}}{2c}\coth\left(\frac{\sqrt{D}}{2}t+\operatorname{arcoth}\left(\frac{2cf^0-b}{\sqrt{D}}\right)\right) & \text{if } f^0>R \\ \frac{b+\sqrt{D}}{2c} & \text{if } f^0=R \\ \frac{b}{2c}+\frac{\sqrt{D}}{2c}\tanh\left(\frac{\sqrt{D}}{2}t+\operatorname{artanh}\left(\frac{2cf^0-b}{\sqrt{D}}\right)\right) & \text{if } 0<f^0<R \end{cases}$$

for all $t \geqslant 0$.

Proof
Note that f is well-defined. Thus the assertion follows by verification. □

Based on the preceding lemma we are able to establish several estimates for the total number of droplets.

Theorem 3.3
Suppose that $0<\varphi_*\leqslant\varphi(v)\leqslant\varphi^*<\infty$ for $v\in L_1^+$ and assume that

(i) $0<K_*\leqslant K(x,y)$ for a.a. $(x,y)\in(0,x_0]^2$;
(ii) there exist $\bar\gamma>0$ and $\sigma\geqslant 0$ such that

$$\int_0^x \left(1-\frac{y}{x}\right)\gamma(x,y)\,\mathrm{d}y\leqslant\bar\gamma x^\sigma \quad \text{for a.a. } 0<x\leqslant x_0 \tag{16}$$

(iii) there exist $z_0\in(0,x_0]$ and $\varepsilon>0$ such that

$$\int_0^{x+y}\beta(x+y,z)\,\mathrm{d}z\leqslant\frac{2-P(x,y)-\varepsilon}{1-P(x,y)} \tag{17}$$

for a.a. $(x,y)\in(0,x_0]^2$ with $x+y\leqslant z_0$.

Then there exist $c>0$, depending on φ and the kernels only, and $\mu:=\mu(\sigma)\geqslant 0$ such that

$$\|u(t;u^0)\|\leqslant c(\|u^0\|+\|u^0\|^\mu),\quad t\geqslant 0$$

where $\mu>0$ if $\sigma>0$.

Proof
First observe that $J(u^0)=\mathbb{R}^+$ by Theorem 2.9 and (17). Next we integrate $(**)$ with respect to x. From Lemma 2.6 and (16) we deduce

$$\int_0^{x_0} L_\mathrm{b}[u](x)\,\mathrm{d}x\leqslant\bar\gamma M_\sigma(t),\quad t\geqslant 0$$

If $\sigma>0$ then we choose $\alpha\in(0,\min\{1,\sigma\})$; otherwise we put $\alpha:=0$. Then Hölder's inequality yields for $\beta:=(\sigma-\alpha)/(1-\alpha)$ and $t\geqslant 0$

$$M_\sigma(t)\leqslant M_1(0)^\alpha M_\beta(t)^{1-\alpha}\leqslant x_0^{\sigma-\alpha}M_1(0)^\alpha M_0(t)^{1-\alpha}$$

By defining sets A and B and the function v as in the proof of Theorem 2.9 we obtain with the aid of Lemma 2.6, (17) and conservation of mass

$$\begin{aligned}\int_0^{x_0} L_\mathrm{c}[u,u](x)\,\mathrm{d}x &\leqslant \frac{1}{2}\int_A [P(x,y)-2+(1-P(x,y))v(x+y)]K(x,y)u(x)u(y)\,\mathrm{d}(x,y)\\ &\quad+\frac{1}{2}\int_B |P(x,y)-2+(1-P(x,y))v(x+y)|K(x,y)u(x)u(y)\,\mathrm{d}(x,y)\\ &\leqslant -\frac{\varepsilon}{2}\int_A K(x,y)u(x)u(y)\,\mathrm{d}(x,y)+(m_\beta+2)\frac{\|K\|_\infty}{z_0}M_1(0)M_0(t)\end{aligned} \tag{18}$$

for $t \geqslant 0$. Moreover, since (i) holds, it follows for $C := (0, x_0]^2 \setminus A$ that

$$\begin{aligned}
-\frac{\varepsilon}{2}\int_A K(x,y)u(x)u(y)\,\mathrm{d}(x,y) &= -\frac{\varepsilon}{2}\int_{(0,x_0]^2} K(x,y)u(x)u(y)\,\mathrm{d}(x,y)\\
&\quad + \frac{\varepsilon}{2}\int_C K(x,y)u(x)u(y)\,\mathrm{d}(x,y)\\
&\leqslant -\frac{\varepsilon}{2}K_* M_0(t)^2 + \varepsilon\frac{\|K\|_\infty}{z_0}M_1(0)M_0(t)
\end{aligned}$$

for $t \geqslant 0$. This and (18) yield

$$\int_0^{x_0} L_{\mathrm{c}}[u,u](x)\,\mathrm{d}x \leqslant -a_1 M_0(t)^2 + a_2 M_1(0)M_0(t), \quad t\geqslant 0$$

with $a_i > 0$ being independent of $t \geqslant 0$ and $u^0 \in L_1^+$. Similarly, the estimate

$$\int_0^{x_0} L_{\mathrm{s}}[u,u](x)\,\mathrm{d}x \leqslant (m_\beta - 2)\frac{\|K\|_\infty}{x_0}M_1(0)M_0(t), \quad t\geqslant 0$$

can be achieved. Putting these facts together and recalling that φ is bounded from below and above, we arrive at the differential inequality for M_0

$$\dot{M}_0(t) \leqslant b_1 M_1(0)^\alpha M_0(t)^{1-\alpha} - b_2 M_0(t)^2 + b_3 M_1(0)M_0(t), \quad t \geqslant 0 \tag{19}$$

where the constants $b_i > 0$ depend neither on $t \geqslant 0$ nor on $u^0 \in L_1^+$.

The function $h(z) := az^{1-\alpha} - bz^2$, $z \geqslant 0$, with $a, b > 0$ and $0 \leqslant \alpha < 1$, has a global maximum at $z^* = (a(1-\alpha)/2b)^{1/(1+\alpha)}$ and thus

$$h(z) \leqslant c(\alpha, b)a^{2/(1+\alpha)}, \quad z \geqslant 0$$

for some $c(\alpha, b) > 0$. Therefore, we can estimate the right-hand side of (19) to obtain

$$\dot{M}_0(t) \leqslant b_4 M_1(0)^{2\alpha/(1+\alpha)} - \frac{b_2}{2}M_0(t)^2 + b_3 M_1(0)M_0(t), \quad t\geqslant 0 \tag{20}$$

with $b_4 > 0$. Since $\coth|_{\mathbb{R}^+}$ is decreasing and tanh is bounded by 1, Lemma 3.2 applied to (20) implies either

$$M_0(t) \leqslant M_0(0), \quad t \geqslant 0$$

if $M_0(0)$ is sufficiently large, or, otherwise,

$$\begin{aligned}
M_0(t) &\leqslant c_1 M_1(0) + c_2\sqrt{M_1(0)^2 + M_1(0)^{2\alpha/(1+\alpha)}}\\
&\leqslant c_3(M_1(0) + M_1(0)^{\alpha/(1+\alpha)}), \quad t \geqslant 0
\end{aligned}$$

with constants $c_i > 0$ depending only on the b_j's. Since $M_1(0) \leqslant x_0 M_0(0)$ the assertion follows by setting $\mu(\sigma) := \alpha/(1+\alpha)$. □

Remarks 3.4
(i) If $\sigma = 0$ then assumption (ii) of Theorem 3.3 is redundant in view of Hypothesis (H_2). However, Theorem 3.3 gives a uniform bound for the total number of droplets while, in the case where $\sigma > 0$, it even leads to stability of the trivial solution for the semiflow generated on L_1^+. Likewise, if no spontaneous breakage occurs, i.e. $\gamma \equiv 0$, then one can choose $\sigma > 0$ arbitrarily, of course. In this case, if, in addition, only binary breakage is considered, it is easily seen that

$$\begin{aligned}\dot{M}_0(t) &= \varphi(t) \int_0^{x_0} L_c[u,u](x)\,dx \\ &= -\frac{1}{2}\varphi(t)\int_0^{x_0}\int_0^{x_0-x} P(x,y)K(x,y)u(x)u(y)\,dy\,dx \\ &\leqslant 0\end{aligned}$$

for $t \geqslant 0$, that is, the total number of droplets decreases with time, and it remains constant if also $P \equiv 0$.
(ii) In liquid–liquid dispersions, conditions like (16) seem to be quite natural if droplets are assumed to be spherical. For further explanation and special kernels satisfying the hypotheses of Theorem 3.3 we refer to Example 3.10.
(iii) Condition (17) is fulfilled if either $P \equiv 1$ or $P(x,y) \geqslant \varepsilon > 0$ for a.a. $x + y \leqslant z_0$ and only binary breakage occurs (see (14)).

As a consequence of the two preceding theorems we have

Corollary 3.5
If the assumptions of Theorem 3.1 or of Theorem 3.3 hold then

$$u(\cdot\,; u^0) \in BC^1(\mathbb{R}^+, L_1)$$

Proof
Since in both cases $\|u(t)\| \leqslant c(u^0)$, $t \geqslant 0$, for some $c(u^0) > 0$ we obtain from $(**)$ and Lemma 2.1

$$\|\dot{u}(t)\| \leqslant c\|\varphi\|_\infty (c(u^0) + c(u^0)^2) < \infty, \quad t \geqslant 0$$

□

Remark 3.6
Any bounded solution u for $(**)$ in $C^k(\mathbb{R}^+, L_1)$ for some $k \geqslant 1$, belongs automatically to $BC^k(\mathbb{R}^+, L_1)$ provided φ is bounded if $k = 1$ and $\varphi \equiv const$ for $k > 1$. This can be shown inductively.

In contrast to the preceding considerations we now assume the breakage action to be rather effective for small droplets, in the sense that sufficiently many droplets are produced by rupture.

Namely, we suppose that the spontaneous breakage frequency is bounded below by a positive constant, which means that also small droplets decay spontaneously at a minimal rate.

Consequently, the following theorems do not apply to the case of pure collisional breakage, i.e. $\gamma \equiv 0$.

Theorem 3.7
Suppose that $0 < \varphi_* \leqslant \varphi(v) \leqslant \varphi^* < \infty$ for $v \in L_1^+$ and that

$$\int_0^x \left(1 - \frac{y}{x}\right) \gamma(x, y)\, \mathrm{d}y \geqslant \gamma_* > 0 \quad \text{for a.a. } 0 < x \leqslant x_0 \tag{21}$$

Then there exists $c_0 > 0$, depending only on φ and the kernels, such that

$$\liminf_{t \nearrow t^+(u^0)} \|u(t; u^0)\| \; \geqslant \; c_0, \quad u^0 \in L_1^+ \setminus \{0\}$$

Proof
Due to Theorem 2.2 we may assume $t^+(u^0) = \infty$. Integrating $(**)$ with respect to x and applying Lemma 2.6 we obtain the differential inequality

$$\dot{M}_0(t) \geqslant \varphi_* \gamma_* M_0(t) - \frac{m_\beta + 2}{2} \varphi^* \|K\|_\infty M_0(t)^2, \quad t \geqslant 0$$

where we, additionally, used the positivity of u and

$$\int_0^{x_0} L_{\mathrm{s}}[u, u](x)\, \mathrm{d}x \geqslant 0 \;, \quad t \geqslant 0$$

Since $\coth|_{\mathbb{R}^+}$ is bounded from below by 1 and $\tanh(z) \nearrow 1$ for $z \nearrow \infty$, the assertion is a consequence of Lemma 3.2 with

$$c_0 := \frac{2\varphi_* \gamma_*}{\varphi^* \|K\|_\infty (m_\beta + 2)}$$

□

For certain collision kernels the assumptions on the breakage frequency can be weakened as follows:

Theorem 3.8
Suppose that $0 < \varphi_* \leqslant \varphi(v) \leqslant \varphi^* < \infty$ for $v \in L_1^+$. Also assume that there are γ_*, $K^* > 0$ such that

$$\int_0^x \left(1 - \frac{y}{x}\right) \gamma(x, y)\, \mathrm{d}y \geqslant \gamma_* x \quad \text{for a.a. } 0 < x \leqslant x_0$$

and

$$K(x, y) \leqslant K^*(x + y) \quad \text{for a.a. } (x, y) \in (0, x_0]^2$$

Then there exists $c_0 > 0$, depending only on φ and the kernels, such that

$$\liminf_{t \nearrow \infty} \|u(t; u^0)\| \geqslant c_0$$

for any $u^0 \in L_1^+ \setminus \{0\}$.

Proof

Theorem 2.9 gives $J(u^0)=\mathbb{R}^+$ and thus, due to Lemma 2.6,

$$\dot{M}_0(t)\geqslant\varphi_*\gamma_*M_1(0)-\varphi^*K^*(m_\beta+2)M_1(0)M_0(t),\quad t\geqslant 0$$

i.e.

$$M_0(t)\geqslant(M_0(0)-c_0)\mathrm{e}^{-\varphi^*K^*(m_\beta+2)M_1(0)t}+c_0,\quad t\geqslant 0$$

where $c_0>0$. □

Corollary 3.9

If the hypotheses of Theorem 3.7 or of Theorem 3.8 hold, then the trivial solution is not stable for the semiflow on L_1^+.

Example 3.10

To illustrate our preceding statements we now consider some special kernels. For simplicity, we assume $P\equiv 1$ which means that two colliding droplets always coalesce and therefore, there is no collisional breakage. As a consequence, the solutions exist for all time. We take spontaneous breakage kernels of the form

$$\gamma(x,y):=a(x)b(x,y),\quad 0<y<x\leqslant x_0$$

where $a(x)$ is the rate at which droplets of mass x break and $b(x,y)$ represents the distribution of fragments formed from a splitting droplet of mass x. Conservation of mass leads to the normalization

$$\int_0^x yb(x,y)\,\mathrm{d}y=x,\quad 0<x\leqslant x_0 \tag{22}$$

Moreover, the quantity

$$v(x):=\int_0^x b(x,y)\,\mathrm{d}y,\quad 0<x\leqslant x_0$$

gives the expected number of droplets when x breaks. Thus $v(x)\geqslant 2$ if $a(x)>0$. If binary breakage is considered, i.e.

$$b(x,y)=b(x,x-y),\quad 0<y<x\leqslant x_0 \tag{23}$$

then (22) implies $v(x)=2$, $0<x\leqslant x_0$. The case where $a(x)$ has no zeros corresponds to *complete breakage*.

(I) Consider the case of *limited breakage* (cf. Reference [3]) which simply means that there exists a stable droplet size $x_s\in(0,x_0)^\dagger$, depending mainly on impeller diameter and speed, such that droplets which are smaller than x_s have a zero breakage frequency, i.e.

$$a(x)=0\quad\text{for a.a. } 0<x\leqslant x_s$$

†In some settings (see Reference [3]) the stable droplet size can be characterized by *Weber's relation*

$$x_s=10^{-4}\pi\rho^{-0.8}\sigma^{1.8}(\omega^2D^{4/3})^{-1.8}$$

where σ and ρ are the surface tension and the density of the dispersed phase, respectively, ω is the impeller speed and D denotes the impeller diameter.

Then

$$\int_0^x \left(1-\frac{y}{x}\right)\gamma(x,y)\,\mathrm{d}y = a(x)(v(x)-1) \leqslant \frac{\|a\|_\infty \|v\|_\infty}{x_s} x$$

for a.a. $0 < x \leqslant x_0$ provided that a and v are bounded. Thus (16) holds.

(II) Suppose *complete breakage* in a strong form such that

$$a(x) \geqslant \underline{a} > 0 \quad \text{for a.a. } 0 < x \leqslant x_0$$

Since in this case $v(x) \geqslant 2$, Theorem 3.7 is valid because of

$$\int_0^x \left(1-\frac{y}{x}\right)\gamma(x,y)\,\mathrm{d}y \geqslant \underline{a} > 0 \quad \text{for a.a. } 0 < x \leqslant x_0$$

(III) A 'power law breakup' (see e.g. References [4, 3], or [10]) is of the form

$$a(x) := hx^\alpha, \quad b(x,y) := f(x)y^\zeta \quad \text{for } 0 < y < x \leqslant x_0$$

with $0 \geqslant \zeta > -2$, $h > 0$, and where the function f is determined by (22), i.e.

$$f(x) := (\zeta+2)x^{-(1+\zeta)}, \quad 0 < x \leqslant x_0$$

In view of (23) binary breakage corresponds to $\zeta = 0$. The underlying idea is that if droplets are assumed to be spherical, the mass x of a droplet is proportional to d^3 where d denotes its diameter. Accordingly, if the mechanism of breakage is independent of the droplet involved or depends either on the diameter itself, or on the surface area, or on the volume of the droplet, α is given by $0, \frac{1}{3}, \frac{2}{3}$ or 1, and, analogously, for ζ.

It is now easy to check that Hypothesis (H_2) is satisfied provided $\alpha \geqslant 0$ and $0 \geqslant \zeta > -1$. Moreover, since

$$v(x) = \frac{\zeta+2}{\zeta+1}, \quad 0 < x \leqslant x_0$$

we have

$$\int_0^x \left(1-\frac{y}{x}\right)\gamma(x,y)\,\mathrm{d}y = \frac{h}{\zeta+1}x^\alpha, \quad 0 < x \leqslant x_0$$

Taking now coalescence kernels of the form

$$K(x,y) := A + B(x+y)^\sigma + C(xy)^\tau, \quad 0 < x, y \leqslant x_0$$

with $A, B, C \geqslant 0$ and $\sigma, \tau \geqslant 0$ we can distinguish the following cases:

(i) If $\alpha = 0$, which means that the breakage rate does not depend on the droplet size, then Theorem 3.7 implies that the trivial solution $u = 0$ is unstable. Furthermore, if also $A > 0$ then, for any initial distribution $u^0 \in L_1^+$, the total number of droplets remains bounded, thanks to Theorem 3.3.

(ii) If $\alpha \in (0, 1], A = 0$ and $\sigma, \tau \geqslant 1$, then the trivial solution is also unstable since Theorem 3.8 holds.

(iii) If $\alpha > 0$ and $A > 0$, then we have stability of $u = 0$ and, given any initial distribution $u^0 \in L_1^+$, the total number of droplets has an upper bound, thanks to Theorem 3.3.

(IV) Consider the case of 'parabolic breakup' (cf. Reference [10]) which means that

$$a(x) := hx^{\eta}, \quad b(x, y) := g(x) y^{\omega}(x - y)$$

with $h > 0$ and g given by

$$g(x) := (\omega + 2)(\omega + 3) x^{-(\omega+2)}, \quad 0 < x \leqslant x_0$$

for $1 \geqslant \omega > -2$. Here, $\omega = 1$ means binary breakage. Then Hypothesis (H_2) holds for $\eta \geqslant 0$, $1 \geqslant \omega > -1$, and the expected number of fragments formed by rupture is

$$v(x) = \frac{\omega + 3}{\omega + 1}, \quad 0 < x \leqslant x_0$$

Therefore,

$$\int_0^x \left(1 - \frac{y}{x}\right) \gamma(x, y) \, \mathrm{d}y = \frac{2h}{\omega + 1} x^{\eta}, \quad 0 < x \leqslant x_0$$

and, taking the same coalescence kernel, we can distinguish the same cases as done in (III).

REFERENCES

1. Fasano A, Rosso F. A new model for the dynamics of dispersions in a batch reactor: theory and numerical simulation. In *Lectures on Applied Mathematics: Proceedings of the Symposium on the Occasion of Karl-Heinz Hoffmann's 60th Birthday*, Bungartz HJ, Hoppe R, Zenger C (eds), Munich, June 30–July 1, 1999, Springer: Berlin, Heidelberg, New York, 2000; 123–141.
2. Panoussopoulos K. Separation of crude oil-water emulsions: experimental techniques and models. *Ph.D. Thesis*, ETH Zurich, 1998.
3. Valentas K, Bilous O, Amundson NR. Analysis of breakage in dispersed phase systems. *I&E C Fundamentals* 1966; **5**:271–279.
4. Valentas K, Bilous O, Amundson NR. Breakage and coalescence in dispersed phase systems. *I&E C Fundamentals* 1966; **5**:533–542.
5. Cheng Z, Redner S. Scaling theory of fragmentation. *Physical Review Letters* 1988; **60**(24):2450–2453.
6. Laurençot P, Wrzosek D. The Discrete Coagulation Equations with Collisional Breakage. *Journal of Statistical Physics* 2001:**104**(1–2):193–253.
7. Amann H. Gewöhnliche Differentialgleichungen (2. Auflage.). de Gruyter: Berlin, New York, 1995.
8. Hille E, Phillips RS. *Functional Analysis and Semigroups*, vol. XXXI (Revised edn.), American Mathematical Society Colloquium Publications: Providence, RI, 1957.
9. McGrady ED, Ziff RM. Shattering transition in fragmentation. *Physical Review Letters* 1987; **58**(9): 892–895.
10. Vigil RD, Ziff RM. On the stability of coagulation–fragmentation population balances. *Journal of Colloid and Interface Science* 1989; **133**:257–264.

