Available online at www.sciencedirect.com
SCIENCE DIRECT®

Nonlinear Analysis 58 (2004) 121 – 142

Nonlinear Analysis

www.elsevier.com/locate/na

# The discrete diffusive coagulation–fragmentation equations with scattering

Christoph Walker

*Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland*

Received 18 November 2003; accepted 4 May 2004

**Abstract**

We consider the discrete coagulation–fragmentation equations with diffusion presupposing a maximal cluster size. Such a feature requires a new interaction mechanism in order to prevent occurrence of too large clusters being produced by coagulation. Existence of a unique solution for this model is proven and long-time behaviour is studied in situations, where equilibria are explicitly known.


*Keywords:* Coagulation–fragmentation equations; Maximal cluster size; Volume scattering; Collisional breakage

## 1. Introduction

In this paper, we study the particular case of the discrete coagulation–fragmentation equations with diffusion, when a maximal cluster size is presupposed. This feature requires a new interaction mechanism opposing the increase of clusters due to coagulation. The idea is that colliding particles with cumulative size beyond the maximal size may merge, but result in a highly unstable cluster which immediately scatters into particles with size less than or equal to the maximal size. Such a scattering mechanism was introduced in [17] for non-diffusive continuous coagulation–fragmentation processes describing the dynamics of two-phase liquids, and was then developed further in [31,32]. We adopt here this idea to discrete processes taking into account movement of particles due to diffusion. Moreover, we also consider the possibility of collisional

*E-mail address:* cwalker@math.unizh.ch (C. Walker).

0362-546X/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2004.05.003

breakage. Even though contemplated in physical literature (see [12,13], or [34]), it has hardly been investigated mathematically so far (however, see [24,31–33]).

We assume clusters to be multiples of an elementary identical unit, and we denote by $M$ the maximal cluster size. Representing the number of $i$-clusters at time $t$ and position $x$ by $u_i = u_i(t,x)$—more precisely,

$$\int_B u_i(t,x)\,\mathrm{d}x$$

accounts for the number of clusters of size $i$ contained in the space region $B$ at time $t$—the evolution of clusters undergoing coagulation and fragmentation can be described by the reaction–diffusion equations

$$\begin{aligned} \dot{u}_i - d_i\Delta u_i &= \varphi(u)g_i(x,u) && \text{in } \Omega,\ \ t>0, \\ \partial_\nu u_i &= 0 && \text{on} \partial\Omega,\ \ t>0, \qquad \text{(CF)} \\ u_i(0) &= u_i^0 && \text{in } \Omega \end{aligned}$$

for $1 \leqslant i \leqslant M$, where $u := (u_1,\dots,u_M)$. Here $\Omega$ is a bounded and smooth domain in $\mathbb{R}^n$, $\nu$ is its outward unit normal vector, and $d_i > 0$ are the diffusion coefficients. The reaction terms $g_i(x,u)$ are defined for $1 \leqslant i \leqslant M$ and $x \in \Omega$ as

$$\begin{aligned} g_i(x,u) := {} & \sum_{j=i+1}^{M} \gamma_{j,i}(x)u_j - u_i \sum_{j=1}^{i-1} \frac{j}{i}\,\gamma_{i,j}(x) + \frac{1}{2}\sum_{j=1}^{i-1} K_{j,i-j}(x)P_{j,i-j}(x)u_j u_{i-j} \\ & + \frac{1}{2}\sum_{j=i+1}^{M}\sum_{k=1}^{j-1} K_{k,j-k}(x)Q_{k,j-k}(x)\beta^{\mathrm{c}}_{j,i}(x)u_k u_{j-k} \\ & - u_i \sum_{j=1}^{M-i} (P_{i,j}(x)+Q_{i,j}(x))K_{i,j}(x)u_j \\ & + \frac{1}{2}\sum_{j=M+1}^{2M}\sum_{k=j-M}^{M} K_{k,j-k}(x)\beta^{\mathrm{s}}_{j,i}(x)u_k u_{j-k} - u_i \sum_{j=M-i+1}^{M} K_{i,j}(x)u_j \end{aligned}$$

with the convention that a sum is defined as zero if the upper summation index is smaller than the lower one. The coefficients $\gamma_{i,j}, 1 \leqslant j < i \leqslant M$, represent the rate at which an $i$-cluster splits into a cluster of size $j$, so the first two sums in the definition of $g_i(x,u)$ give the gain and loss of $i$-clusters due to fragmentation. The next three sums describe the possible interactions of two colliding clusters $i$ and $j$ with cumulative size $i+j$ less than or equal to $M$. They either coalesce with probability $P_{i,j}$, or a shattering of these particles occurs with probability $Q_{i,j}$, or just nothing happens meaning that the clusters remain unchanged. This demands

$$0 \leqslant P_{i,j}(x) + Q_{i,j}(x) \leqslant 1, \quad 1 \leqslant i+j \leqslant M, \quad x \in \Omega. \tag{1}$$

The rate of collision of two clusters $i$ and $j$ is denoted by $K_{i,j}$. For high-energy collisions of clusters $i$ and $j$ with $i+j \leqslant M$, $\beta^{\mathrm{c}}_{i+j,k}$ stands for the expected number of fragments of size $k \in \{1,\dots,i+j-1\}$. Furthermore, the last two sums in the definition of $g_i(x,u)$ reflect the scattering process. For $i+j \in \{M+1,\dots,2M\}$, $\beta^{\mathrm{s}}_{i+j,k}$ gives the expected number of daughter clusters of size $k \in \{1,\dots,M\}$. Both collisional breakage and scattering are assumed to be mass preserving, that is, for all $x \in \Omega$ it holds

$$\sum_{j=1}^{i-1} j\beta^{\mathrm{c}}_{i,j}(x) = i, \quad 2 \leqslant i \leqslant M \quad \text{and} \quad \sum_{j=1}^{M} j\beta^{\mathrm{s}}_{i,j}(x) = i, \quad M < i \leqslant 2M. \tag{2}$$

Finally, also a new feature in our model is the efficiency factor $\varphi(u)$ enhancing or depressing the dynamics, while the mechanical structure of the processes are described by the kernels $\gamma_{i,j}, \beta^{\mathrm{c}}_{i,j}, \beta^{\mathrm{s}}_{i,j}, K_{i,j}, P_{i,j}$, and $Q_{i,j}$. For instance, a possible choice of $\varphi$ is

$$\varphi(u) := \Phi\left(\sum_{i=1}^{M} i \int_\Omega u_i(x)\,\mathrm{d}x, \sum_{i=1}^{M} \int_\Omega u_i(x)\,\mathrm{d}x\right),$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}^+$ is a given function. This means that $\varphi(u)$ is related to the total mass and the total number of particles.

As mentioned before, the model above is an adaptation of the continuous model without diffusion considered in [31,33], which, on the other hand, is based on the model proposed in [17] (see also [9,26]). For a treatment of the continuous model with diffusion we refer to [32]. To the best of our knowledge, discrete coagulation –fragmentation processes including the scattering phenomenon have never been considered in literature so far, whereas literature on discrete models without scattering is quite extensive. For the latter case with diffusion we refer to [8,14,15,19,21,23,25,35], and the references therein.

Clearly, presupposing a maximal cluster size simplifies the problem enormously—as long as not global existence is concerned for cluster size-dependent diffusion coefficients, as we shall see.

This paper is organized as follows: in Section 2 we introduce the notation that will be used throughout. Section 3 is dedicated to existence results. Finally, Section 4 then deals with certain aspects of large time behaviour. It should be remarked that our finite-dimensional problem (CF) always admits infinitely many equilibria—that is, spatially homogeneous steady states—without *any* assumptions on the kernels restricting the physics, except for being independent of the space variable $x$. Three different situations will then be analysed, for which convergence towards equilibrium can be shown.

## 2. Notations and conventions

Let us introduce some notations, which will be used in the sequel. Given any interval $J$ in $\mathbb{R}$ we put $\dot{J} := J \setminus \{0\}$. Furthermore, $\mathscr{L}(E,F)$ stands for the set of all continuous and linear operators from a Banach space $E$ into another Banach space $F$ equipped with the topology of uniform convergence on bounded subsets.

In the following, $\Omega$ will always denote a bounded and smooth domain in $\mathbb{R}^n$. Given $1 \leqslant p \leqslant \infty$ and $\mu \geqslant 0$ we denote by $W^\mu_p := W^\mu_p(\Omega)$ the usual Sobolev–Slobodeckii space of order $\mu$, and we also put $L_p := L_p(\Omega)$. Observe that $L_p = W_p^0$ and

$$W_p^\mu \hookrightarrow W_q^\alpha, \quad \mu > \alpha \geqslant 0, \quad \mu - n/p > \alpha - n/q.$$

Moreover, we define

$$W_{p,\mathscr{B}}^\mu := \begin{cases} \{u \in W_p^\mu; \partial_\nu u = 0\}, & \mu > 1 + 1/p, \\ W_p^\mu, & 0 \leqslant \mu < 1 + 1/p. \end{cases}$$

Then it is well known (see [30]) that for $1 < p < \infty$

$$(L_p, W_{p,\mathscr{B}}^2)_{\theta,p} \doteq W_{p,\mathscr{B}}^{2\theta}, \quad 2\theta \in (0,2) \setminus \{1, 1+1/p\}$$

and

$$[L_p, W_{p,\mathscr{B}}^2]_{1/2} \doteq W_{p,\mathscr{B}}^1,$$

where $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ denote the real and the complex interpolation functor, respectively.

For the next few basic properties of the Laplace operator subject to Neumann boundary conditions we refer to [2,28]. We denote by $A_1$ the closure of $-\Delta|_{C_{\mathscr{B}}^2(\Omega)}$ in $L_1$, where $C_{\mathscr{B}}^2(\Omega) := \{u \in C^2(\Omega); \partial_\nu u = 0\}$, and for $1 < p < \infty$ we set $A_p u := -\Delta u$ for $u \in W_{p,\mathscr{B}}^2$. Then, $-A_p$ is for each $p \in [1,\infty)$ the generator of a positive, compact analytic semigroup $\{e^{-tA_p}; t \geqslant 0\}$ of contractions in $L_p$. It holds

$$A_1|_{L_p \cap \mathsf{dom}(A_1)} = A_p, \quad 1 < p < \infty$$

and

$$e^{-tA_1}|_{L_p} = e^{-tA_p}, \quad t \geqslant 0, \quad 1 < p < \infty.$$

This justifies to set $A := A_1$ in the sequel. From the estimates

$$\|e^{-tA}\|_{\mathscr{L}(L_p,L_q)} \leqslant c_T t^{-(n/2)(1/p-1/q)}, \quad 0 < t \leqslant T, \quad 1 \leqslant p < q \leqslant \infty$$

and

$$\|Ae^{-tA}\|_{\mathscr{L}(L_p)} \leqslant c_T t^{-1}, \quad 0 < t \leqslant T, \quad 1 \leqslant p < \infty,$$

it follows by interpolation

$$\|e^{-tA}\|_{\mathscr{L}(L_p,W_{q,\mathscr{B}}^\alpha)} \leqslant c_T t^{-(n/2)(1/p-1/q)-\alpha/2}, \quad 0 < t \leqslant T$$

for $\alpha \in [0,2] \setminus \{1+1/q\}$ and $1 \leqslant p \leqslant q < \infty$, where $q > 1$. We also have

$$\|e^{-tA}\|_{\mathscr{L}(W_{p,\mathscr{B}}^\alpha,W_{p,\mathscr{B}}^\mu)} \leqslant c_T t^{-(\mu-\alpha)/2}, \quad 0 < t \leqslant T$$

for $1 < p < \infty$ and $0 \leqslant \alpha \leqslant \mu \leqslant 2$ with $\alpha, \mu \neq 1 + 1/p$.

For fixed $d_i > 0, 1 \leqslant i \leqslant M$, it is then obvious that all of these properties carry over to the generator

$$-\mathbb{A} := -\mathbb{A}(d_1,\ldots,d_M) := \mathsf{diag}[-d_1A,\ldots,-d_MA]$$

of the semigroup

$$e^{-t\mathbb{A}} := \mathsf{diag}[e^{-td_1A},\ldots,e^{-td_MA}]$$

in $L_p := L_p(\Omega,\mathbb{R}^M)$. To simplify the notation we put again $W_p^\mu := W_p^\mu(\Omega,\mathbb{R}^M)$ and $W_{p,\mathscr{B}}^\mu := W_{p,\mathscr{B}}^\mu(\Omega,\mathbb{R}^M)$ since there will no confusion arise in the sequel whether the spaces are scalar- or vector-valued. Moreover, $L_p^+$ denotes the positive cone of $L_p$, and $\|\cdot\|_p$ is the norm in $L_p$.

## 3. Existence, uniqueness, and other properties

Throughout we use the notations of the last section and we assume the following hypotheses to be satisfied:

(H$_1$) $\varphi : L_2 \to \dot{\mathbb{R}}^+$ is uniformly Lipschitz continuous and bounded on bounded subsets of $L_2 = L_2(\Omega, \mathbb{R}^M)$.
(H$_2$) Each of the maps $\gamma_{i,j}, \beta^{\mathrm{c}}_{i,j}, \beta^{\mathrm{s}}_{i,j}, K_{i,j}, P_{i,j}$, and $Q_{i,j}$ is $\rho$-Hölder continuous from $\bar{\Omega}$ into $\mathbb{R}^+$ for some $\rho > 0$.
(H$_3$) Collisional breakage and scattering are mass preserving, that is, $\beta^{\mathrm{c}}_{i,j}$ and $\beta^{\mathrm{s}}_{i,j}$ satisfy (2).
(H$_4$) For all $x \in \Omega$ it holds $R_{i,j}(x) = R_{j,i}(x)$ for $R \in \{K, P, Q\}$, and $P$ and $Q$ satisfy (1).
(H$_5$) For each $i \in \{1, \ldots, M\}$ it holds $d_i > 0$.

We rewrite the discrete coagulation–fragmentation equations (CF) as a semilinear Cauchy problem of the form

$$\dot{u} + \mathbb{A}u = f(u), \qquad t > 0,$$

$$u(0) = u^0, \tag{$*$}$$

where $\mathbb{A} := \mathbb{A}(d_1, \ldots, d_M), u = (u_1, \ldots, u_M), u^0 = (u_1^0, \ldots, u_M^0)$, and $f$ is given by $f(u) := \varphi(u) g(u)$ with $g$ denoting the Nemytskii operator induced by $(g_1, \ldots, g_M)$, that is

$$g(u)(x) := (g_1(x, u(x)), \ldots, g_M(x, u(x))), \qquad x \in \Omega, \qquad u : \Omega \to \mathbb{R}^M.$$

If $J \subset \mathbb{R}^+$ denotes a perfect interval containing 0, we mean by a *mild* $L_p$-solution to problem ($*$) a function $u \in C(J, L_p)$ satisfying the integral equation

$$u(t) = \mathrm{e}^{-t\mathbb{A}} u^0 + \int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(u(s))\, \mathrm{d}s, \qquad t \in J.$$

A *strong* $L_p$-solution to ($*$) is a function

$$u \in C(J, L_p) \cap C^1(\dot{J}, L_p) \cap C(\dot{J}, W^2_{p,\mathscr{B}})$$

satisfying ($*$) pointwise.

In particular, given any mild (strong) $L_p$-solution $u$ to ($*$), each component $u_i$ is then a mild (strong) $L_p$-solution to the $i$th equation of (CF).

Let us observe that, for $u = (u_1, \ldots, u_M), r = (r_1, \ldots, r_M) \in \mathbb{R}^M$, and $x \in \Omega$, the identity

$$\sum_{i=1}^{M} r_i g_i(x, u) = \sum_{i=2}^{M} \sum_{j=1}^{i-1} \left( r_j - \frac{j}{i} r_i \right) \gamma_{i,j}(x) u_i$$

$$+ \frac{1}{2} \sum_{2 \leqslant i+j \leqslant M} \Bigg\{ r_{i+j} P_{i,j}(x) - (r_i + r_j)(P_{i,j}(x) + Q_{i,j}(x))$$

$$+Q_{i,j}(x)\sum_{k=1}^{i+j-1} r_k\beta^{\mathrm{c}}_{i+j,k}(x)\Bigg\}K_{i,j}(x)u_iu_j$$

$$+\frac{1}{2}\sum_{M<i+j\leqslant 2M}\left(\sum_{k=1}^{M} r_k\beta^{\mathrm{s}}_{i+j,k}(x)-r_i-r_j\right)K_{i,j}(x)u_iu_j \tag{3}$$

holds, which we will use in the following.

We then can prove the following theorem on existence and uniqueness of solutions.

**Theorem 1.** *Let $2\leqslant p<\infty$ and $n<2p$. Then, given any $u^0\in L_p^+$, problem $(*)$ possesses a unique maximal mild solution $u:=u(\cdot;u^0)\in C(J(u^0),L_p^+)$, where $J(u^0)$ denotes the maximal interval of existence. In addition, it holds*

$$\sum_{i=1}^{M} i\int_\Omega u_i(t)\,\mathrm{d}x=\sum_{i=1}^{M} i\int_\Omega u_i^0\,\mathrm{d}x,\qquad t\in J(u^0) \tag{4}$$

*and the solution $u$ has the regularity*

$$u\in C(\dot{J}(u^0),W^\mu_{p,\mathscr{B}}),\qquad \mu\in[0,2-n/p)\setminus\{1+1/p\}, \tag{5}$$

*where the dot of $\dot{J}(u^0)$ can be skipped provided that $u^0\in W^\mu_{p,\mathscr{B}}$. Furthermore, if $n<4p/3$ then, for each $\mu\in[0,2]\setminus\{1+1/p\}$ and $u^0\in W^\mu_{p,\mathscr{B}}$*

$$u\in C(J(u^0),W^\mu_{p,\mathscr{B}})\cap C(\dot{J}(u^0),W^2_{p,\mathscr{B}})\cap C^1(\dot{J}(u^0),L_p) \tag{6}$$

*is a strong solution to $(*)$. Finally, if $d_1=\cdots=d_M$ then $J(u^0)=\mathbb{R}^+$ and*

$$\|u(t)\|_p\leqslant c\|u^0\|_p,\qquad t\geqslant 0 \tag{7}$$

*for some $c>0$ independent of $p$.*

**Proof.** We perform the proof in several steps.

(i) First, recall hypothesis $(\mathrm{H}_1)$ and observe that

$$\|g(v)-g(w)\|_{p/2}\leqslant c(1+\|v\|_p+\|w\|_p)\|v-w\|_p,\qquad v,w\in L_p, \tag{8}$$

whence $f:L_p\to L_{p/2}$ is uniformly Lipschitz continuous on bounded subsets of $L_p$. Also note that

$$\|\mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(L_{p/2},L_p)}\leqslant c_T t^{-n/2p},\qquad 0<t\leqslant T.$$

Then, since $t\mapsto t^{-n/2p}$ is integrable on $(0,T)$ for each $T>0$, standard arguments entail that problem $(*)$ admits for each $u^0\in L_p$ a unique maximal mild solution $u:=u(\cdot;u^0)\in C(J(u^0),L_p)$. Moreover, $u(\cdot;u^0)$ depends continuously on the initial value $u^0$ in the sense that, given any $T\in J(u^0)$, there is a neighbourhood $U$ of $u^0$ in $L_p$ such that $[0,T]\subset J(v^0)$ for each $v^0\in U$ and

$$u(\cdot;v^0)\to u(\cdot;u^0)\quad\text{in } C([0,T],L_p)\quad\text{as } v^0\to u^0.$$

Furthermore, (4) follows from

$$\int_\Omega \mathrm{e}^{-td_iA}v\,\mathrm{d}x=\int_\Omega v\,\mathrm{d}x,\qquad v\in L_p(\Omega),\qquad t\geqslant 0, \tag{9}$$

which is consequence of the Neumann boundary conditions, and from

$$\sum_{i=1}^{M} ig_i(x,w)=0, \quad w\in\mathbb{R}^M, \quad x\in\Omega. \tag{10}$$

(ii) Next, due to

$$\|\mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(L_{p/2},W^{\mu}_{p,\mathscr{B}})} \leqslant c_T t^{-n/2p-\mu/2}, \quad 0<t\leqslant T, \quad \mu\neq 1+1/p,$$

we deduce

$$\left(t\mapsto\int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(u(s))\,\mathrm{d}s\right)\in C(J(u^0),W^{\mu}_{p,\mathscr{B}})$$

for $\mu\in[0,2-n/p)\setminus\{1+1/p\}$ since $f(u)\in C(J(u^0),L_{p/2})$. This implies (5).

(iii) Assume now that $n<4p/3$ so that we find $\sigma\in(n/2p,2-n/p)\setminus\{1+1/p\}$. Fix $\varepsilon\in\dot{J}(u^0)$ and put $J_\varepsilon:=(J(u^0)-\varepsilon)\cap\mathbb{R}^+$. Then $u^\varepsilon:=u(\cdot+\varepsilon;u^0)$ is a mild solution to the linear problem

$$\dot{v}+\mathbb{A}v=b^\varepsilon(t), \quad t\in\dot{J}_\varepsilon,$$
$$v(0)=u^\varepsilon(0), \tag{11}$$

where $b^\varepsilon:=f(u^\varepsilon)$. Since $u^\varepsilon\in C(J_\varepsilon,W^{\sigma}_{p,\mathscr{B}})$ by (5) and $\sigma>n/2p$, the multiplication result of [3, Theorem 4.1] implies $b^\varepsilon\in C(J_\varepsilon,W^{\alpha}_{p,\mathscr{B}})$ for some $\alpha>0$ sufficiently small (recall that the kernels are Hölder continuous with respect to $x\in\Omega$). Applying then [5, II.Theorem 1.2.2] we derive that

$$u^\varepsilon\in C^1(\dot{J}_\varepsilon,L_p)\cap C(\dot{J}_\varepsilon,W^2_{p,\mathscr{B}})$$

is a strong solution to (11), due to the fact that mild solutions to linear problems are unique. Let then $\varepsilon$ tend to zero in order to conclude that

$$u\in C^1(\dot{J}(u^0),L_p)\cap C(\dot{J}(u^0),W^2_{p,\mathscr{B}})$$

is a strong solution to problem $(*)$.

Furthermore, if $u^0\in W^{\mu}_{p,\mathscr{B}}$ for some $\mu\in[2-n/p,2]\setminus\{1+1/p\}$, we have as above $f(u)\in C(J(u^0),W^{\alpha}_{p,\mathscr{B}})$. Therefore,

$$\left(t\mapsto\int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(u(s))\,\mathrm{d}s\right)\in C(J(u^0),W^{\mu}_{p,\mathscr{B}})$$

by virtue of

$$\|\mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(W^{\alpha}_{p,\mathscr{B}},W^{\mu}_{p,\mathscr{B}})} \leqslant c_T t^{-(\mu-\alpha)/2}, \quad 0<t\leqslant T, \quad \alpha\neq 1+1/p.$$

It follows that (6) holds true.

(iv) We now show positivity of the solution. If $n<4p/3$ and $u^0\in W^2_{p,\mathscr{B}}\cap L_p^+$ we see that

$$u_i\in C(J(u^0),W^2_{p,\mathscr{B}})\hookrightarrow C(J(u^0),L_\infty)$$

is for each $i\in\{1,\dots,M\}$ a strong solution to a problem of the form

$$\dot{u}_i-d_i\Delta u_i=h_i(u,u)-u_iH_i(u), \quad \partial_\nu u_i=0, \quad u_i(0)\geqslant 0,$$

where $h_i$ and $H_i$ are functions satisfying $h_i \geqslant 0$ on $(\mathbb{R}^+)^M \times (\mathbb{R}^+)^M$ and $H_i \geqslant 0$ on $(\mathbb{R}^+)^M$, respectively. Thus, standard arguments entail that $u_i(t;u^0) \geqslant 0$ in $\Omega$ for each $i \in \{1,\dots,M\}$ and $t \in J(u^0)$.

If $u^0 \in L_p^+$ and still $n < 4p/3$, use the continuous dependence on the initial value and the density of $W^2_{p,\mathscr{B}} \cap L_p^+$ in $L_p^+$ (see [5, V.Proposition 2.7.1]) in order to deduce that $u_i(t;u^0) \geqslant 0$ a.e. in $\Omega$ for each $i \in \{1,\dots,M\}$ and $t \in J(u^0)$.

We proceed with a bootstrapping argument. Temporarily, the mild solution to $(*)$ for $u^0 \in L_p^+$ will be denoted by $u^{(p)} \in C(J_p(u^0), L_p)$. We then say that $P(\zeta)$ holds true if

$$u^{(p)}(t) \in L_p^+ \quad \text{for} \quad t \in J_p(u^0) \quad \text{and} \quad n < \zeta p.$$

Next, we claim that $P(\zeta)$ implies $P(1+\zeta/2)$ provided that $\zeta \in [\frac{4}{3}, 2)$. For, let $P(\zeta)$ hold true for some $\zeta \in [\frac{4}{3}, 2)$ and assume $\zeta \leqslant n/p < 1 + \zeta/2$. Then there exists $\varepsilon > 0$ sufficiently small such that $2 - 2n/p > -\zeta + 2\varepsilon$. Put $\mu := 2 - n/p - \varepsilon$ and $q := n/(\zeta - \varepsilon)$. In particular, $\mu - n/p > -n/q$ so that, given $u^0 \in W^2_{p,\mathscr{B}} \cap L_p^+$, we have

$$u^{(p)} \in C(J_p(u^0), W^\mu_{p,\mathscr{B}}) \hookrightarrow C(J_p(u^0), L_q)$$

according to (5). But then $u^{(q)} \supset u^{(p)}$ and hence $u^{(p)}(t) \in L_p^+$ for $t \in J_p(u^0)$ since $P(\zeta)$ holds true by assumption. Using again the continuous dependence on the initial value and the density of $W^2_{p,\mathscr{B}} \cap L_p^+$ in $L_p^+$, we deduce that $P(\zeta)$ indeed implies $P(1+\zeta/2)$ for $\zeta \in [\frac{4}{3}, 2)$.

But we already know that $P(\frac{4}{3})$ holds true. Hence, we inductively obtain that $P(\zeta_j)$ holds true, where $\zeta_j := 1 + \zeta_{j-1}/2$ for $j \geqslant 1$ with $\zeta_0 := \frac{4}{3}$. Since $\zeta_j \nearrow 2$ we thus have proved that $u(t;u^0) \in L_p^+$ for $t \in J(u^0)$ whenever $u^0 \in L_p^+$ and $n < 2p$.

(v) Finally, recalling the definition of a mild solution and the fact that $-\mathbb{A}$ generates a contraction semigroup, (10) entails in the case $d_1 = \cdots = d_M$

$$\left\| \sum_{i=1}^M i u_i(t;u^0) \right\|_p \leqslant \sum_{i=1}^M i \|u_i^0\|_p, \qquad t \in J(u^0),$$

whence the positivity of $u_i(t;u^0)$ yields (7) and $J(u^0) = \mathbb{R}^+$. This completes the proof. □

**Remark 2.** It is out of our knowledge how to prove global existence if the diffusion coefficients depend on the cluster size. For instance, the method developed in [35, Lemma 2.2] seems not to work due to the scattering term.

## 4. Asymptotic behaviour

We now focus our attention on long-time behaviour of the solutions obtained in the previous section. For the remainder, we assume that the diffusion coefficients are independent of cluster size, that is,

$$d := d_1 = \cdots = d_M > 0.$$

In the following, for $u^0 \in L_p^+$ given with $2 \leqslant p < \infty$ and $n < 2p$, we denote by $u = u(\cdot; u^0) \in C(\mathbb{R}^+, L_p^+)$ the unique mild solution to problem $(*)$. Let us remark that we require cluster size-independent coefficients mainly to guarantee global existence.

We first need an auxiliary result in the spirit of [7].

**Proposition 3.** *Let $2 \leqslant p < \infty$ and $n < 2p$. Given $T > 0$ put*

$$\Phi(f)(t) := \int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(s)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T, \quad f \in L_\infty((0,T), L_{p/2}).$$

*Then $\Phi \in \mathscr{L}(L_\infty((0,T), L_{p/2}), C([0,T], L_p))$ maps bounded sets into compact sets.*

**Proof.** For $v \in L_{p/2}$ and $t, h > 0$ it follows from [27, Theorem 1.2.4(b)] that

$$\begin{aligned}\|\mathrm{e}^{-(t+h)\mathbb{A}} v - \mathrm{e}^{-t\mathbb{A}} v\|_p &= \left\| \mathbb{A} \int_0^h \mathrm{e}^{-s\mathbb{A}} \mathrm{e}^{-t\mathbb{A}} v \,\mathrm{d}s \right\|_p \\ &\leqslant \|\mathbb{A}\mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(L_p)} \int_0^h \|\mathrm{e}^{-s\mathbb{A}}\|_{\mathscr{L}(L_{p/2}, L_p)} \,\mathrm{d}s \|v\|_{p/2},\end{aligned}$$

whence

$$\|\mathrm{e}^{-(t+h)\mathbb{A}} - \mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(L_{p/2}, L_p)} \leqslant c_T \frac{h^{1-n/2p}}{t},$$
$$t > 0, \quad h \geqslant 0, \quad t + h \leqslant T + 1. \tag{12}$$

Use then this inequality instead of (4) in [7] and replace (3) of the latter by

$$\|\mathrm{e}^{-t\mathbb{A}}\|_{\mathscr{L}(L_{p/2}, L_p)} \leqslant c_T t^{-n/2p}, \quad 0 < t \leqslant T.$$

Then the proofs of [7, Lemmas 1(ii), 2] carry over to our situation. This implies the statement. □

As an immediate consequence we deduce the following corollary.

**Corollary 4.** *Let $2 \leqslant p < \infty$ and $n < 2p$. Given any $u^0 \in L_p^+$, any sequence $t_m \nearrow \infty$, and any $T > 0$ there exists a subsequence $(t_{m_k})$ and $\bar{u} \in C([0,T], L_p^+)$ such that*

$$u(\cdot + t_{m_k}; u^0) \to \bar{u} \quad \text{in } C([0,T], L_p).$$

*Moreover, it holds*

$$\bar{u}(t) = \mathrm{e}^{-t\mathbb{A}} \bar{u}(0) + \int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(\bar{u}(s))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T$$

*and*

$$\sum_{i=1}^M i \int_\Omega \bar{u}_i(t)\,\mathrm{d}x = \sum_{i=1}^M i \int_\Omega u_i^0 \,\mathrm{d}x, \quad 0 \leqslant t \leqslant T. \tag{13}$$

**Proof.** For $t_m \geqslant 1$ put $u^m := u(\cdot + t_m - 1)$. By Theorem 1, the sequence $(f(u^m))_m$ is bounded in $L_\infty((0,T+1),L_{p/2})$ so that we can extract a subsequence $(m_k)$ in order to deduce that $(\Phi(f(u^{m_k}))_{m_k}$ converges in $C([0,T+1],L_p^+)$ due to Proposition 3. Moreover, invoking (12), the Arzelà–Ascoli theorem entails that we can extract a further subsequence $(m_k')$ such that $(\mathrm{e}^{-t\mathbb{A}}u^{m_k'}(0))_{m_k'}$ converges in $C([1,T+1],L_p^+)$. Therefore, $(u(\cdot + t_{m_k'}))_{m_k'}$ converges in $C([0,T],L_p^+)$. The statements are then consequences of

$$u(t+t_{m_k'};u^0) = \mathrm{e}^{-t\mathbb{A}}u(t_{m_k'}) + \int_0^t \mathrm{e}^{-(t-s)\mathbb{A}} f(u(s+t_{m_k'}))\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T,$$

of (8) combined with Gronwall's lemma, and of (4). □

**Remark 5.** Given $2 \leqslant p < \infty$ with $n < 2p$, Corollary 4 states in fact that the $\omega$-limit set $\omega_p(u^0)$ in $L_p$, defined by

$$\omega_p(u^0) := \{v \in L_p;\ \text{there exists } t_m \nearrow \infty \text{ with } u(t_m;u^0) \to v \text{ in } L_p\}$$

is for each $u^0 \in L_p^+$ non-empty. Obviously, it holds, in addition, $\omega_p(u^0) \subset L_p^+$ and

$$\sum_{i=1}^M i \int_\Omega v_i \,\mathrm{d}x = \sum_{i=1}^M i \int_\Omega u_i^0 \,\mathrm{d}x, \quad v \in \omega_p(u^0).$$

Temporarily assume that all kernels are independent of $x \in \Omega$. Considering then the ordinary differential equation

$$\dot z = g(z), \quad t > 0, \quad z(0) = z^0,$$

in $\mathbb{R}^M$, it is easily seen that

$$Y_\varrho := \left\{ z \in \mathbb{R}^M; z_i \geqslant 0, \sum_{i=1}^M i z_i = \varrho \right\}$$

is for each $\varrho > 0$ a compact, convex, and positively invariant set. According to [4, Satz 22.13] there exists $u^\varrho \in Y_\varrho$ satisfying $g(u^\varrho) = 0$. Due to the Neumann boundary conditions, the original problem $(*)$ thus always has infinitely many equilibria—-that is, spatially homogeneous steady states—provided that the non-negative kernels satisfy $(\mathrm{H}_3)$ and $(\mathrm{H}_4)$ and do not depend on $x \in \Omega$.

Based on Corollary 4, we consider now some special cases for which equilibria are explicitly known.

### *4.1. Dominating coagulation*

Throughout this subsection we assume that hypotheses $(\mathrm{H}_1)$–$(\mathrm{H}_4)$ hold. Additionally, we suppose that there is no fragmentation, that is, $\gamma \equiv 0$, and that only binary shattering and binary scattering occurs, i.e., for each $x \in \Omega$ it holds

$$\beta^{\mathrm{c}}_{i,j}(x) = \beta^{\mathrm{c}}_{i,i-j}(x), \quad 1 \leqslant j < i \leqslant M, \tag{14}$$

as well as

$$\beta^{\mathrm{s}}_{i,j}(x) = \beta^{\mathrm{s}}_{i,i-j}(x), \quad 1 \leqslant i - M \leqslant j \leqslant M \tag{15}$$

and

$$\beta^{\mathrm{s}}_{i,j}(x) = 0, \quad 1 \leqslant j < i - M \leqslant M. \tag{16}$$

Observe that $(H_3)$ then implies for each $x \in \Omega$

$$\sum_{j=1}^{i-1} \beta^{\mathrm{c}}_{i,j}(x) = 2, \quad 2 \leqslant i \leqslant M \quad \text{and} \quad \sum_{j=i-M}^{M} \beta^{\mathrm{s}}_{i,j}(x) = 2, \quad M < i \leqslant 2M. \tag{17}$$

In particular, this entails that $(0, \ldots, 0, a) \in \mathbb{R}^M$ is for each $a \geqslant 0$ an equilibrium of problem $(*)$ since $\beta^{\mathrm{s}}_{2M,M} \equiv 2$.

We also require the technical assumptions [1]

$$P_{i,i}(x) K_{i,i}(x) > 0, \quad 1 \leqslant i \leqslant [M/2], \quad x \in \Omega \tag{18}$$

and that for each $i \in \{[M/2] + 1, \ldots, M - 1\}$ there is $r \in \{2i - M, \ldots, i - 1\}$ with

$$K_{i,i}(x) \beta^{\mathrm{s}}_{2i,r}(x) > 0, \quad x \in \Omega. \tag{19}$$

Then we can prove the following result.

**Theorem 6.** *Suppose $\gamma \equiv 0$ and that (14)–(19) are satisfied. For $2 \leqslant p < \infty$ and $n < 2p$ let $u^0 \in L_p^+$. Then, given any sequence $t_m \nearrow \infty$, it holds*

$$u_i(\cdot + t_m; u^0) \to 0 \quad \text{in} \quad C(\mathbb{R}^+, L_p), \quad 1 \leqslant i \leqslant M - 1.$$

*Moreover, for $T > 0$ there exists a subsequence $(t_{m_k})$ and $\bar{u}^{\,0} \in L_p^+(\Omega)$ with*

$$\int_\Omega \bar{u}^{\,0} \, \mathrm{d}x = \frac{1}{M} \sum_{i=1}^{M} i \int_\Omega u_i^0 \, \mathrm{d}x$$

*such that*

$$u_M(\cdot + t_{m_k}; u^0) \to \bar{u} \quad \text{in} \quad C([0, T], L_p),$$

*where $\bar{u}(t) := \mathrm{e}^{-tdA} \bar{u}^{\,0}, t \geqslant 0$, is the unique solution to*

$$\dot{v} - d\Delta v = 0, \quad \partial_\nu v = 0, \quad v(0) = \bar{u}^{\,0}.$$

**Proof.** For $u_i = u_i(\cdot; u^0)$ define

$$N(t) := \sum_{i=1}^{M} \int_\Omega u_i(t, x) \, \mathrm{d}x, \quad t \geqslant 0 \tag{20}$$

so that (3), (9), and (14)–(16) imply

$$N(t) + \frac{1}{2} \sum_{2 \leqslant i+j \leqslant M} \int_0^t \varphi(u) \int_\Omega P_{i,j} K_{i,j} u_i u_j \, \mathrm{d}x \, \mathrm{d}s = N(0), \quad t \geqslant 0. \tag{21}$$

[1] For $m \in \mathbb{N}$, $[m]$ denotes the Gauss brackets of $m$, i.e., $[m]$ is defined as $m/2$ if $m$ is even and it is defined as $(m-1)/2$ otherwise.

Let $T>0$ be arbitrary. Due to Corollary 4 we may choose a subsequence $(t_{m_k})$ and $\bar u\in C([0,T],L_p^+)$ such that

$$u^{m_k}:=u(\cdot+t_{m_k};u^0)\to\bar u \quad \text{in} \quad C([0,T],L_p).$$

According to $(H_2)$ and (21) we obtain

$$0\leqslant\sum_{2\leqslant i+j\leqslant M}\int_0^T\varphi(\bar u)\int_\Omega P_{i,j}K_{i,j}\bar u_i\bar u_j\,\mathrm{d}x\,\mathrm{d}s$$

$$=\lim_k\sum_{2\leqslant i+j\leqslant M}\int_{t_{m_k}}^{T+t_{m_k}}\varphi(u)\int_\Omega P_{i,j}K_{i,j}u_iu_j\,\mathrm{d}x\,\mathrm{d}s=0.$$

Eq. (18) entails then

$$\bar u_i(t)=0,\quad 1\leqslant i\leqslant[M/2],\quad 0\leqslant t\leqslant T. \tag{22}$$

Next, we claim that if, for some $l\in\{[M/2],\dots,M-2\}$,

$$\bar u_i(t)=0,\quad 1\leqslant i\leqslant l,\quad 0\leqslant t\leqslant T,$$

then $\bar u_{l+1}(t)=0$ for $0\leqslant t\leqslant T$. Indeed, (19) guarantees that we can choose some $r\in\{2(l+1)-M,\dots,l\}$ such that

$$K_{l+1,l+1}(x)\beta^{\mathrm{s}}_{2(l+1),r}(x)>0,\quad x\in\Omega$$

and hence

$$0=\int_0^T\int_\Omega f_r(\bar u)\,\mathrm{d}x\,\mathrm{d}s=\frac12\int_0^T\varphi(\bar u)\int_\Omega\sum_{j=M+1}^{M+r}\sum_{k=j-M}^{M}K_{k,j-k}\beta^{\mathrm{s}}_{j,r}\bar u_k\bar u_{j-k}\,\mathrm{d}x\,\mathrm{d}s$$

$$\geqslant\frac12\int_0^T\varphi(\bar u)\int_\Omega K_{l+1,l+1}\beta^{\mathrm{s}}_{2(l+1),r}|\bar u_{l+1}|^2\,\mathrm{d}x\,\mathrm{d}s\geqslant 0,$$

where the first equality stems from (9) and Corollary 4. Therefore, we deduce from (22) by induction that

$$\bar u_i(t)=0,\quad 1\leqslant i\leqslant M-1,\quad 0\leqslant t\leqslant T.$$

In particular, this implies

$$f_M(\bar u(t))=\varphi(\bar u(t))K_{M,M}\left(\frac12\beta^{\mathrm{s}}_{2M,M}-1\right)|\bar u_M(t)|^2=0,\quad 0\leqslant t\leqslant T$$

due to $\beta^{\mathrm{s}}_{2M,M}\equiv 2$. Corollary 4 then yields

$$\bar u_M(t)=\mathrm{e}^{-tdA}\bar u_M(0),\quad 0\leqslant t\leqslant T. \qquad \square$$

**Remark 7.** (a) It seems to be reasonable to suppose that $\bar u$ from the previous theorem is independent of spatial coordinates. However, we were not able to prove it for lack of a suitable a priori estimate for $\nabla u_M$. Under the assumptions of the following subsections we will obtain corresponding estimates.

(b) A result similar to the one of Theorem 6 was obtained in [8, Theorem 3.1, Remark 3.2] for diffusive discrete coagulation processes (without fragmentation) in the case $M=\infty$. More precisely, the—not necessarily unique—solution $u$ constructed in [8, Theorem 3.1] satisfies $\|u_i(t)\|_\infty \to 0$ for all $i \geqslant 1$.

### 4.2. Dominating fragmentation

In this subsection, we again assume hypotheses $(H_1)$–$(H_4)$ to hold. Moreover, we suppose that, for each $x \in \Omega$,

$$a_{i,j}(x) := Q_{i,j}(x)\left\{\sum_{k=1}^{i+j-1} \beta^{c}_{i+j,k}(x) - 2\right\} - P_{i,j}(x) \geqslant 0, \quad 2 \leqslant i+j \leqslant M \tag{23}$$

and

$$b_i(x) := \sum_{j=1}^{M} \beta^{s}_{i,j}(x) - 2 \geqslant 0, \quad M < i \leqslant 2M. \tag{24}$$

Observe that (24) does not restrict the physical scope of applications since the sum in (24) represents the number of daughter clusters being produced from a splitting $i$-cluster. Also note that (23) implies $a_{1,1} \equiv P_{1,1} \equiv 0$ due to $\beta^{c}_{2,1} \equiv 2$ (see $(H_3)$). Therefore, particles of size 1 do not interact implying, in particular, that $(a,0,\dots,0) \in \mathbb{R}^M$ with $a \geqslant 0$ is an equilibrium for problem $(*)$.

**Theorem 8.** *Let $u^0 \in L^+_\infty$. In addition to (23) and (24) suppose that either*

$$K_{i,i}(x)a_{i,i}(x) > 0, \quad 2 \leqslant i \leqslant [M/2], \quad K_{1,j}(x)a_{1,j}(x) > 0, \quad 2 \leqslant j \leqslant M-1 \tag{25}$$

*and*

$$K_{i,i}(x)b_{2i}(x) > 0, \quad [M/2] < i \leqslant M, \quad K_{1,M}(x)b_{M+1}(x) > 0 \tag{26}$$

*for all $x \in \bar{\Omega}$, or that for each $i \in \{2,\dots,M\}$ there exists $j \in \{1,\dots,i-1\}$ with*

$$\gamma_{i,j}(x) > 0, \quad x \in \bar{\Omega}. \tag{27}$$

*Then, given any sequence $t_m \nearrow \infty$ and $p \in [1,\infty)$, it holds*

$$u_1(\cdot + t_m; u^0) \to \frac{1}{|\Omega|}\sum_{i=1}^{M} i \int_\Omega u_i^0 \,\mathrm{d}x \quad \text{in } C(\mathbb{R}^+, L_p)$$

*and*

$$u_i(\cdot + t_m; u^0) \to 0 \quad \text{in } C(\mathbb{R}^+, L_p), \quad 2 \leqslant i \leqslant M.$$

**Proof.** Define $N$ again by (20) and observe

$$N(0) + \sum_{i=1}^{M} \int_0^t \int_\Omega f_i(u)\,\mathrm{d}x\,\mathrm{d}s = N(t) \leqslant MN(0), \quad t \geqslant 0. \tag{28}$$

Moreover, define $c_i(x) \geqslant 0$ by

$$c_i(x) := \sum_{j=1}^{i-1} \left(1 - \frac{j}{i}\right) \gamma_{i,j}(x), \quad 2 \leqslant i \leqslant M, \quad x \in \bar{\Omega}$$

so that, due to (3),

$$\sum_{i=1}^{M} f_i(u) = \varphi(u) \sum_{i=2}^{M} c_i u_i + \frac{1}{2}\, \varphi(u) \sum_{2 \leqslant i+j \leqslant M} K_{i,j} a_{i,j} u_i u_j$$
$$+ \frac{1}{2}\, \varphi(u) \sum_{M < i+j \leqslant 2M} K_{i,j} b_{i+j} u_i u_j. \tag{29}$$

If (25) and (26) hold true, then it follows from (28) and (29)

$$\int_0^\infty \varphi(u) \int_\Omega u_1 u_j \,\mathrm{d}x\,\mathrm{d}s \leqslant c(u^0), \quad 2 \leqslant j \leqslant M. \tag{30}$$

On the other hand, (27) together with (28) and (29) yield

$$\int_0^\infty \varphi(u) \int_\Omega u_j \,\mathrm{d}x\,\mathrm{d}s \leqslant c(u^0), \quad 2 \leqslant j \leqslant M,$$

whence (30), since (7) and $u^0 \in L_\infty^+$ imply $u_i \in L_\infty(\mathbb{R}^+, L_\infty(\Omega))$. Therefore, we have, due to (30) and $\beta_{2,1}^{\mathrm{c}} \equiv 2$, for each $T > 0$

$$\begin{aligned}
\int_0^T \int_\Omega u_1 f_1(u) \,\mathrm{d}x\,\mathrm{d}s = {} & \int_0^T \varphi(u) \int_\Omega \sum_{j=2}^{M} \gamma_{j,1} u_j u_1 \,\mathrm{d}x\,\mathrm{d}s \\
& + \frac{1}{2} \int_0^T \varphi(u) \int_\Omega \sum_{j=2}^{M} \sum_{k=1}^{j-1} K_{k,j-k} Q_{k,j-k} \beta_{j,1}^{\mathrm{c}} u_k u_{j-k} u_1 \,\mathrm{d}x\,\mathrm{d}s \\
& - \int_0^T \varphi(u) \int_\Omega u_1^2 \sum_{j=1}^{M-1} (P_{1,j} + Q_{1,j}) K_{1,j} u_j \,\mathrm{d}x\,\mathrm{d}s \\
& + \frac{1}{2} \int_0^T \varphi(u) \int_\Omega \sum_{j=M+1}^{2M} \sum_{k=j-M}^{M} K_{k,j-k} \beta_{j,1}^{\mathrm{s}} u_k u_{j-k} u_1 \,\mathrm{d}x\,\mathrm{d}s \\
& - \int_0^T \varphi(u) \int_\Omega u_1^2 K_{1,M} u_M \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant {} & c(u^0) \sum_{j=2}^{M} \int_0^T \varphi(u) \int_\Omega u_j u_1 \,\mathrm{d}x\,\mathrm{d}s \\
& + \int_0^T \varphi(u) \int_\Omega \left(\frac{1}{2}\, \beta_{2,1}^{\mathrm{c}} - 1\right) K_{1,1} Q_{1,1} u_1^3 \,\mathrm{d}x\,\mathrm{d}s \\
\leqslant {} & c(u^0)
\end{aligned}$$

with $c(u^0) > 0$ independent of $T > 0$. Take then $u_1$ as a test function for

$$\dot{u}_1 - d\Delta u_1 = f_1(u), \quad t > 0, \quad u_1(0) = u_1^0,$$

in order to obtain

$$\frac{1}{2}\int_\Omega |u_1(T)|^2\,\mathrm{d}x + d\int_0^T\int_\Omega |\nabla u_1|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$= \frac{1}{2}\int_\Omega |u_1^0|^2\,\mathrm{d}x + \int_0^T\int_\Omega u_1 f_1(u)\,\mathrm{d}x\,\mathrm{d}s \leqslant c(u^0), \tag{31}$$

where $c(u^0)$ does not depend on $T > 0$. Given $t_m \nearrow \infty$ and $p \in [2,\infty)$ large we can choose a subsequence $(t_{m_k})$ and $\bar{u} \in C([0,T], L_p^+)$ satisfying (13) such that

$$u^{m_k} := u(\cdot + t_{m_k}; u^0) \to \bar{u} \quad \text{in } C([0,T], L_p), \tag{32}$$

due to Corollary 4. Estimate (31) entails in particular

$$\int_0^T\int_\Omega |\nabla u_1^{m_k}|^2\,\mathrm{d}x\,\mathrm{d}s = \int_{t_{m_k}}^{T+t_{m_k}}\int_\Omega |\nabla u_1|^2\,\mathrm{d}x\,\mathrm{d}s \to 0$$

and hence, by (32)

$$\nabla u_1^{m_k} \to \nabla \bar{u}_1 = 0 \quad \text{in } L_2((0,T), L_2(\Omega, \mathbb{R}^n)). \tag{33}$$

On the other hand, (28) and (29) warrant $\bar{u}_i(t) = 0$ for $0 \leqslant t \leqslant T$ and $2 \leqslant i \leqslant M$ in both of cases (25)–(27). Hence (33) and (13) imply

$$\bar{u}_1(t) \equiv \frac{1}{|\Omega|}\sum_{i=1}^M i\int_\Omega u_i^0\,\mathrm{d}x, \quad 0 \leqslant t \leqslant T.$$

Since $T > 0$ and $p < \infty$ were arbitrary, the assertion follows. □

**Remark 9.** In the case of pure fragmentation and $M = \infty$ it has been proven in [21, Corollary 4.3] that

$$u_1(t) \to \frac{1}{|\Omega|}\sum_{i=1}^\infty i\int_\Omega u_i^0\,\mathrm{d}x \quad \text{in} \quad L_1(\Omega)$$

and

$$u_i(t) \to 0 \quad \text{in } L_1(\Omega), \quad i \geqslant 2.$$

### 4.3. *The detailed balance condition*

The purpose of this subsection is to study a very particular case of the coagulation–fragmentation equations, namely when the kernels satisfy an extended version of the so-called *detailed balance condition* (see (37)). This condition amounts to assume that the processes under consideration are somehow reversible. For the diffusive case without scattering and without shattering, related results were previously obtained in [15] if discrete processes are considered (see also [22] for the Becker–Döring equations), whereas the continuous equations with diffusion were treated in [18]. For the non-diffusive case we refer to [1,10,11,20,29,33].

We assume throughout this subsection that hypotheses $(H_1)$–$(H_4)$ are satisfied and that

$$\text{the kernels } \gamma_{i,j}, \beta^{c}_{i,j}, \beta^{s}_{i,j}, K_{i,j}, P_{i,j}, \text{ and } Q_{i,j} \text{ are independent of } x \in \Omega. \tag{34}$$

We consider merely binary breakage, that is

$$\gamma_{i,j} = \gamma_{i,i-j}, \quad 1 \leqslant j < i \leqslant M \quad \text{and (14)–(16) hold.} \tag{35}$$

Moreover, we suppose

$$\gamma_{i,1} > 0, \quad 2 \leqslant i \leqslant M \tag{36}$$

and that there exists $H_i > 0$, $1 \leqslant i \leqslant M$, such that

$$\begin{aligned} &\gamma_{i+j,i} H_{i+j} = P_{i,j} K_{i,j} H_i H_j, \quad 2 \leqslant i+j \leqslant M, \\ &\beta^{c}_{i,j} Q_{k,i-k} K_{k,i-k} H_k H_{i-k} = \beta^{c}_{i,k} Q_{j,i-j} K_{j,i-j} H_j H_{i-j}, \quad 1 \leqslant j,k < i \leqslant M, \\ &\beta^{s}_{i,j} K_{k,i-k} H_k H_{i-k} = \beta^{s}_{i,k} K_{j,i-j} H_j H_{i-j}, \quad 1 \leqslant i-M \leqslant j,k \leqslant M. \end{aligned} \tag{37}$$

Let us observe that a possible choice of kernels is as follows.

**Example 10.** Let $\alpha, \xi \in \mathbb{R}$ be arbitrary and suppose that $P_{i,j} = P_{j,i} > 0$ and $Q_{i,j} = q(i+j)$ are given for $1 \leqslant i+j \leqslant M$, where $q$ is a non-negative function with $P_{i,j} + q(i+j) \leqslant 1$. Putting

$$\begin{aligned} &K_{i,j} := K^*(i+j)^{\alpha}, \quad 1 \leqslant i,j \leqslant M, \\ &\gamma_{i,j} := \gamma^* P_{i-j,j} i^{\alpha-\xi} (j(i-j))^{\xi}, \quad 1 \leqslant j < i \leqslant M, \\ &\beta^{c}_{i,j} := i(j(i-j))^{\xi} \left( \sum_{k=1}^{i-1} k^{1+\xi} (i-k)^{\xi} \right)^{-1}, \quad 1 \leqslant j < i \leqslant M, \\ &\beta^{s}_{i,j} := i(j(i-j))^{\xi} \left( \sum_{k=i-M}^{M} k^{1+\xi} (i-k)^{\xi} \right)^{-1}, \quad 1 \leqslant i-M \leqslant j \leqslant M \end{aligned}$$

for some $K^*, \gamma^* > 0$, hypotheses $(H_3)$ and $(H_4)$ as well as (34)–(37) are satisfied with

$$H_i := \frac{\gamma^*}{K^*} i^{\xi}, \quad 1 \leqslant i \leqslant M.$$

For $w \in L_p^+$ introduce

$$V(w) := \sum_{i=1}^{M} \int_{\Omega} w_i(x) \left( \log \frac{w_i(x)}{H_i} - 1 \right) \mathrm{d}x$$

with the convention $r(\log r - 1) := 0$ for $r = 0$, and observe that $V(w)$ is well defined provided $p > 1$ due to the inequality

$$r|\log r| \leqslant c(\varepsilon)(r^{1+\varepsilon} + r^{1-\varepsilon}), \quad r \geqslant 0, \quad \varepsilon > 0. \tag{38}$$

Furthermore, set

$$J(a,b) := \begin{cases} (a-b)(\log a - \log b), & a,b>0, \\ 0, & a=b=0, \\ \infty, & \text{else} \end{cases}$$

and define $D(v) \in \mathbb{R}^+ \cup \{\infty\}$ for $v \in \mathbb{R}^M$ by

$$\begin{aligned} D(v) := &\frac{1}{2} \sum_{2 \leqslant i+j \leqslant M} J(\gamma_{i+j,i} v_{i+j}, P_{i,j} K_{i,j} v_i v_j) \\ &+ \frac{1}{8} \sum_{2 \leqslant i+j \leqslant M} \sum_{k=1}^{i+j-1} J(\beta^{\rm c}_{i+j,k} Q_{i,j} K_{i,j} v_i v_j, \beta^{\rm c}_{i+j,i} Q_{k,i+j-k} K_{k,i+j-k} v_k v_{i+j-k}) \\ &+ \frac{1}{8} \sum_{M < i+j \leqslant 2M} \sum_{k=i+j-M}^{M} J(\beta^{\rm s}_{i+j,k} K_{i,j} v_i v_j, \beta^{\rm s}_{i+j,i} K_{k,i+j-k} v_k v_{i+j-k}). \end{aligned}$$

We first establish an auxiliary result stating in fact that $V$ is a Lyapunov function for $(*)$.

**Proposition 11.** *Let (34)–(37) be satisfied and assume $2 \leqslant p < \infty$ with $n < 2p$. Then, given $u^0 \in L_p^+$, it holds for $u = u(\cdot; u^0)$*

$$V(u(t)) \leqslant V(u(s)), \quad t \geqslant s \geqslant 0. \tag{39}$$

*In addition,*

$$\int_0^\infty \varphi(u) \int_\Omega D(u)\, \mathrm{d}x\, \mathrm{d}t < \infty \tag{40}$$

*and, for $\eta := 2p/(p+1) \in [\frac{4}{3}, 2)$,*

$$\int_0^\infty \left( \int_\Omega |\nabla u_i|^\eta \, \mathrm{d}x \right)^{2/\eta} \mathrm{d}t < \infty, \quad 1 \leqslant i \leqslant M. \tag{41}$$

**Proof.** For $m \geqslant \max \sqrt{H_i}$ define $u^{0,m} = (u_1^{0,m}, \ldots, u_M^{0,m})$ by

$$u_i^{0,m} := \min\{m, \max\{u_i^0, H_i/m\}\}, \quad 1 \leqslant i \leqslant M$$

and observe that $H_i/m \leqslant u_i^{0,m} \leqslant m$ a.e. in $\Omega$ for all $i \in \{1, \ldots, M\}$. It is then straightforward to check that

$$\limsup_m V(u^{0,m}) \leqslant V(u^0). \tag{42}$$

Furthermore, we have $\|u^{0,m}\|_p \leqslant c(u^0)$ with $c(u^0) > 0$ independent of $m$. Hence, Theorem 1 entails that

$$u^m := u(\cdot; u^{0,m}) \in C(\mathbb{R}^+, L_q^+) \cap L_\infty(\mathbb{R}^+, L_\infty), \quad q > \max\{2, 3n/4, p\}$$

is a strong $L_q$-solution with

$$\|u^m(t)\|_p \leqslant c(u^0), \quad t \geqslant 0. \tag{43}$$

In addition, for $T > 0$ arbitrary we may assume

$$u^m \to u(\cdot; u^0) \quad \text{in } C([0,T], L_p) \quad \text{and a.e. in } (0,T) \times \Omega, \tag{44}$$

since $u^{0,m} \to u^0$ in $L_p$. Thus, $u_i^m$ satisfies

$$\dot{u}_i^m - d\Delta u_i^m \geqslant -c_m u_i^m \quad \text{a.e. in } \Omega, \quad t > 0$$

for some $c_m > 0$, whence $u_i^m(t) \geqslant (H_i/m)\mathrm{e}^{-c_m t}$ a.e. in $\Omega$ for $t \geqslant 0$. Due to this we may take $\log u_i/H_i$ as a test function in the $i$th equation of (CF). Then we derive from (3) and (17) after some calculations that, for $t > 0$,

$$V(u^m(t)) + d\sum_{i=1}^M \int_0^t \int_\Omega \frac{1}{u_i^m} |\nabla u_i^m|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \varphi(u^m) \int_\Omega D(u^m)\,\mathrm{d}x\,\mathrm{d}s = V(u^{0,m}).$$

On the other hand, observing that

$$h_r(z) := z\left(\log\frac{z}{r} - 1\right) \geqslant -r = h_r(r), \quad z \geqslant 0, \quad r > 0, \tag{45}$$

it follows

$$V(u^m(t)) \geqslant -|\Omega| \sum_{i=1}^M H_i.$$

Combining these two estimates and invoking (42) we derive

$$\sum_{i=1}^M \int_0^T \int_\Omega \frac{1}{u_i^m} |\nabla u_i^m|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^T \varphi(u^m) \int_\Omega D(u^m)\,\mathrm{d}x\,\mathrm{d}s \leqslant c(u^0), \tag{46}$$

where $c(u^0) > 0$ does neither depend on $m$ nor on $T$. In particular, owing to this estimate, Hölder's inequality, and (43) we have for $\eta := 2p/(p+1)$

$$\begin{aligned} \sum_{i=1}^M \int_0^T \left(\int_\Omega |\nabla u_i^m|^\eta \,\mathrm{d}x\right)^{2/\eta} \mathrm{d}s &\leqslant \sum_{i=1}^M \int_0^T \|u_i^m(s)\|_p \int_\Omega \frac{1}{u_i^m} |\nabla u_i^m|^2 \,\mathrm{d}x\,\mathrm{d}s \\ &\leqslant c(u^0) \end{aligned} \tag{47}$$

with $c(u^0)$ independent of $T$ and $m$. Hence, $(\nabla u_i^m)_m$ is for each $i \in \{1,\ldots,M\}$ a bounded sequence in $L_2((0,T), L_\eta(\Omega,\mathbb{R}^n))$. From (44) we thus conclude that $(\nabla u_i^m)_m$ converges weakly towards $\nabla u_i$ in $L_2((0,T), L_\eta(\Omega,\mathbb{R}^n))$. Since $T > 0$ was arbitrary, (47) implies then (41). Furthermore, Fatou's lemma, (44), and (42) entail that, for $t \geqslant 0$,

$$V(u(t;u^0)) \leqslant \liminf_m V(u^m(t)) \leqslant \liminf_m V(u^{0,m}) \leqslant V(u^0),$$

whence (39) by virtue of $u(t+s;u^0) = u(t; u(s;u^0))$. Finally, since $D$ is lower semi-continuous, Fatou's lemma, (44), and (46) also yield

$$\int_0^T \varphi(u) \int_\Omega D(u)\,\mathrm{d}x\,\mathrm{d}s \leqslant \liminf_m \int_0^T \varphi(u^m) \int_\Omega D(u^m)\,\mathrm{d}x\,\mathrm{d}s \leqslant c(u^0). \qquad \square$$

**Theorem 12.** *Let (34)–(37) be satisfied and assume $2 \leqslant p < \infty$ with $n < 2p$. For $u^0 \in L_p^+$ choose $\alpha \geqslant 0$ uniquely such that*

$$\sum_{i=1}^{M} iH_i\alpha^i = \frac{1}{|\Omega|}\sum_{i=1}^{M} i\int_\Omega u_i^0\,\mathrm{d}x.$$

*Then, given any sequence $t_m \nearrow \infty$, it holds*

$$u_i(\cdot + t_m; u^0) \to H_i\alpha^i \quad \text{in } C(\mathbb{R}^+, L_p), \quad 1 \leqslant i \leqslant M.$$

*Moreover, if $u^0 \in L_\infty^+$, then also*

$$V(u(t;u^0)) \to V(u^\alpha) \quad \text{as} \quad t \to \infty, \tag{48}$$

*where $u^\alpha := (H_1\alpha, \ldots, H_M\alpha^M)$.*

**Proof.** Put $u^m := u(\cdot + t_m; u^0)$ and let $T > 0$ be arbitrary. Due to Corollary 4 we have, up to a subsequence,

$$u^m \to \bar u \quad \text{in } C([0,T], L_p) \quad \text{and a.e. in } \Omega_T := (0,T)\times\Omega \tag{49}$$

for some $\bar u \in C([0,T], L_p^+)$. From Fatou's lemma and (40) we conclude

$$0 \leqslant \int_0^T \varphi(\bar u)\int_\Omega D(\bar u)\,\mathrm{d}x\,\mathrm{d}t \leqslant \liminf_m \int_{t_m}^{T+t_m} \varphi(u)\int_\Omega D(u)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

whence

$$D(\bar u) = 0 \quad \text{a.e. in } \Omega_T, \tag{50}$$

since $\varphi$ has no zeros. Next, (41) warrants, for $1 \leqslant i \leqslant M$ and $\eta := 2p/(p+1)$,

$$\int_0^T \left(\int_\Omega |\nabla u_i^m|^\eta\,\mathrm{d}x\right)^{2/\eta} \mathrm{d}t = \int_{t_m}^{T+t_m} \left(\int_\Omega |\nabla u_i|^\eta\,\mathrm{d}x\right)^{2/\eta} \mathrm{d}t \to 0,$$

so that $(\nabla u_i^m)$ converges towards $\nabla \bar u_i = 0$ in $L_2((0,T), L_\eta(\Omega,\mathbb{R}^n))$ for each $i \in \{1,\ldots,M\}$ according to (49). In particular, $\bar u_i$ does not depend on $x \in \Omega$ and, due to (50), (37), and the continuity of $\bar u$,

$$\gamma_{i+j,i}\bar u_{i+j}(t) = P_{i,j}K_{i,j}\bar u_i(t)\bar u_j(t) = \frac{\gamma_{i+j,i}H_{i+j}}{H_iH_j}\,\bar u_i(t)\bar u_j(t), \quad 2 \leqslant i+j \leqslant M$$

for $0 \leqslant t \leqslant T$. Recalling (36) and the fact that $\bar u$ satisfies (13), it therefore follows $\bar u_i(t) = H_i\alpha^i$ for $t \in [0,T]$ and $i \in \{1,\ldots,M\}$. Hence, it remains to prove (48). For, let $u^0 \in L_\infty^+$. Then, due to (7), the sequence $(u(t_m;u^0))_m$ is bounded in $L_\infty$, so Lebesgue's theorem, (49) with $\bar u$ replaced by $u^\alpha$, and (38) imply (48). □

**Remark 13.** The above proof shows that, in addition, for $\eta := 2p/(p+1)$,

$$u_i(\cdot + t_m; u^0) \to H_i\alpha^i \quad \text{in } L_{2,\mathrm{loc}}(\mathbb{R}^+, W_\eta^1(\Omega)), \quad 1 \leqslant i \leqslant M.$$

We now focus on stability of the equilibria $u^\alpha := (H_1\alpha, \ldots, H_M\alpha^M)$. For that purpose define, for $\varrho > 0$ and $2 \leqslant p \leqslant \infty$,

$$X_{\varrho,p}^+ := \left\{ w \in L_p^+;\ \sum_{i=1}^{M} i\int_\Omega w_i\,\mathrm{d}x = \varrho \right\}$$

and observe that $X^+_{\varrho,p}$ is positively invariant according to Theorem 1 provided that $n<2p$. Moreover, for $1\leqslant p\leqslant\infty$ put

$$d_p(v,w):=\|v-w\|_p+|V(v)-V(w)|.$$

**Remark 14.** Given $\varrho>0$ choose $\alpha:=\alpha(\varrho)>0$ with

$$\sum_{i=1}^{M} iH_i\alpha^i=\frac{\varrho}{|\Omega|}. \tag{51}$$

Then it follows from Theorems 1 and 12 that the equilibrium $u^\alpha$ is a global attractor in $(X^+_{\varrho,\infty},d_p)$ for each $p\in[1,\infty)$.

In order to proceed, we need the following lemma.

**Lemma 15.** *Let $p\geqslant 2$. For $\varrho>0$ choose $\alpha:=\alpha(\varrho)>0$ such that (51) holds. Then $u^\alpha$ is the unique minimizer of $V$ on $X^+_{\varrho,p}$. Moreover, given any minimizing sequence $(w^m)$ of $V$ in $X^+_{\varrho,p}$, it holds $\|w^m-u^\alpha\|_1\to 0$.*

**Proof.** It readily follows that $u^\alpha$ is the unique minimizer of $V$ on the set of all $w\in L_1^+$ satisfying $V(w)<\infty$ and

$$\sum_{i=1}^{M} i\int_\Omega w_i\,\mathrm{d}x=\varrho \tag{52}$$

— and hence also on $X^+_{\varrho,p}$— by observing that

$$V(w)-V(u^\alpha)=\sum_{i=1}^{M}\|h_{u_i^\alpha}(w_i)-h_{u_i^\alpha}(u_i^\alpha)\|_1,$$

where $h_{u_i^\alpha}$ is defined as in (45). Moreover, given any sequence $(w^m)$ in $X^+_{\varrho,p}$ with $\lim_m V(w^m)=V(u^\alpha)$, we may assume for $1\leqslant i\leqslant M$ that $(h_{u_i^\alpha}(w_i^m))$ converges towards $h_{u_i^\alpha}(u_i^\alpha)$ almost everywhere in $\Omega$, whence $(w_i^m)$ converges towards $u_i^\alpha$ almost everywhere in $\Omega$ for $1\leqslant i\leqslant M$ due to the properties of $h_{u_i^\alpha}$. On the other hand, it follows analogously to [18, Lemma 3.1] that, for any measurable subset $E$ of $\Omega$ and any $\lambda\geqslant e^2$,

$$\sum_{i=1}^{M}\int_E w_i^m\,\mathrm{d}x\leqslant\frac{2}{\log\lambda}V(w^m)+2\left(\frac{|\Omega|}{\log\lambda}+\lambda|E|\right)\sum_{i=1}^{M}H_i,\qquad m\in\mathbb{N}.$$

The Dunford–Pettis theorem (see [16, Theorem 4.21.2]) then guarantees the existence of a subsequence $(m_k)$ such that $(w^{m_k})$ converges weakly in $L_1$ towards some $w\in L_1^+$ satisfying (52). Since $V:L_1(\Omega,\mathbb{R}^M)\to\mathbb{R}\cup\{\infty\}$ is weakly sequentially lower semi-continuous due to its convexity and Fatou's lemma, we therefore have

$$V(w)\leqslant\liminf_{m_k}V(w^{m_k})=V(u^\alpha),$$

whence $w=u^\alpha$. Consequently, $(w^{m_k})$ converges towards $u^\alpha$ weakly in $L_1$ and almost everywhere in $\Omega$. This implies the assertion. $\square$

We conclude with the following corollary on stability of the equilibria $u^\alpha$.

**Corollary 16.** *For $\varrho > 0$ choose $\alpha := \alpha(\varrho) > 0$ such that (51) holds.*

(i) *For $2 \leqslant p < \infty$ with $n < 2p$, the equilibrium $u^\alpha$ is stable in $(X^+_{\varrho,p}, d_1)$, that is, given any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(u(t;u^0), u^\alpha) < \varepsilon$ for $t \geqslant 0$, whenever $u^0 \in X^+_{\varrho,p}$ satisfies $d_1(u^0, u^\alpha) < \delta$.*

(ii) *The equilibrium $u^\alpha$ is asymptotically stable in $(X^+_{\varrho,\infty}, d_1)$.*

**Proof.** It follows from Lemma 15 that, for any $\varepsilon > 0$ small, there exists $\sigma(\varepsilon) > 0$ such that $V(w) - V(u^\alpha) \geqslant \sigma(\varepsilon)$ provided $w \in X^+_{\varrho,p}$ with $\|w - u^\alpha\|_1 = \varepsilon$. Hence, [6, Proposition 4.3] and (39) imply (i). Finally, statement (ii) is a consequence of (i) and Remark 14. □


## Acknowledgements

I would like to thank Prof. H. Amann for helpful suggestions to improve this paper.


## References


[1] M. Aizenman, T.A. Bak, Convergence to equilibrium in a system of reacting polymers, Commun. Math. Phys. 65 (1979) 203–230.

[2] H. Amann, Dual semigroups and second order linear elliptic boundary value problems, Israel J. Math. 45 (1983) 225–254.

[3] H. Amann, Multiplication in Sobolev and Besov spaces, in: A. Ambrosetti, A. Marino (Eds.) Nonlinear Analysis, A Tribute in Honour of Giovanni Prodi; Quaderni. Pisa: Scuola Normale Superiore (1991) 27–57.

[4] H. Amann, Gewöhnliche Differentialgleichungen, de Gruyter, Berlin, New York, 1995.

[5] H. Amann, Linear and Quasilinear Parabolic Problems, Abstract Linear Theory, Vol. I, Birkhäuser, Basel, Boston, Berlin, 1995.

[6] J.M. Ball, J.E. Marsden, Quasiconvexity at the boundary, positivity of the second variation and elastic stability, Arch. Ration. Mech. Anal. 86 (1984) 251–277.

[7] P. Barras, J.C. Hassan, L. Véron, Compacité de l'opérateur définissant la solution d'une équation d'évolution non homogène, C. R. Acad. Sci. Paris Sér. I Math. 284 (1977) 799–802.

[8] P. Bénilan, D. Wrzosek, On an infinite system of reaction–diffusion equations, Adv. Math. Sci. Appl. 7 (1) (1997) 351–366.

[9] I. Borsi, Dynamics of liquid–liquid dispersions with unbounded fragmentation kernel, Adv. Math. Sci. Appl. 11 (2) (2001) 571–591.

[10] J. Carr, Asymptotic behaviour of solutions to the coagulation–fragmentation equations. I. The strong fragmentation case, Proc. R. Soc. Edinburgh 121A (1992) 231–244.

[11] J. Carr, F.P. DA Costa, Asymptotic behaviour of solutions to the coagulation–fragmentation equations. II. Weak fragmentation, J. Stat. Phys. 77 (1/2) (1994) 89–123.

[12] Z. Cheng, S. Redner, Scaling theory of fragmentation, Phys. Rev. Lett. 60 (24) (1988) 2450–2453.

[13] Z. Cheng, S. Redner, Kinetics of fragmentation, J. Phys. A: Math. Gen. 23 (1990) 1233–1258.

[14] J.F. Collet, F. Poupaud, Existence of solutions to coagulation–fragmentation systems with diffusion, Transport Theory Stat. Phys. 25 (1996) 503–513.

[15] J.F. Collet, F. Poupaud, Asymptotic behaviour of solutions to the diffusive fragmentation–coagulation system, Physica D 114 (1998) 123–146.

[16] R.E. Edwards, Functional Analysis, Theory and Applications, Dover Publ., New York, 1995.

[17] A. Fasano, F. Rosso, A new model for the dynamics of dispersions in a batch reactor: theory and numerical simulation, in: H.J. Bungartz, R. Hoppe, C. Zenger (Eds.), Lectures on Applied Mathematics: Proceedings of the Symposium on the Occasion of Karl–Heinz Hoffmann's 60th birthday, Munich, June 30–July 1, 1999, Springer, Berlin, Heidelberg, New York, 2000, pp. 123–141.

[18] P. Laurençot, S. Mischler, The continuous coagulation–fragmentation equations with diffusion, Arch. Ration. Mech. Anal. 162 (1) (2002) 45–99.
[19] P. Laurençot, S. Mischler, Global existence for the discrete diffusive coagulation–fragmentation equations in $L^1$, Rev. Mat. Iberoamericana 18 (2002) 731–745.
[20] P. Laurençot, S. Mischler, Convergence to equilibrium for the continuous coagulation–fragmentation equation, Bull. Sci. Math. 127 (2003) 179–190.
[21] P. Laurençot, D. Wrzosek, Fragmentation–diffusion model. Existence of solutions and their asymptotic behaviour, Proc. R. Soc. Edinburgh 128 (1998) 759–777.
[22] P. Laurençot, D. Wrzosek, The Becker–Döring model with diffusion: II. Long time behaviour, J. Differential Equations 148 (1998) 268–291.
[23] P. Laurençot, D. Wrzosek, Coagulation with partial diffusion, ZAMP 50 (1999) 559–573.
[24] P. Laurençot, D. Wrzosek, The discrete coagulation equations with collisional breakage, J. Stat. Phys. 104 (1–2) (2001) 193–253.
[25] P. Laurençot, D. Wrzosek, From the nonlocal to the local discrete diffusive coagulation equations, Math. Methods Appl. Sci. 12 (7) (2002) 1035–1048.
[26] A. Mancini, F. Rosso, A new model for the dynamics of dispersions in a batch reactor: numerical approach, Meccanica 37 (3) (2002) 221–237.
[27] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1983.
[28] F. Rothe, Global solutions of reaction–diffusion systems, in: A. Dold, B. Eckmann (Eds.) Lecture Notes in Mathematics, Vol. 1072, Berlin, Springer, 1984.
[29] I.W. Stewart, P.B. Dubovskii, Approach to equilibrium for the coagulation–fragmentation equation via a Lyapunov functional, Math. Methods Appl. Sci. 19 (1996) 171–183.
[30] G. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 2nd Edition, Johann Ambrosius Barth, Heidelberg, Leipzig, 1995.
[31] C. Walker, Coalescence and breakage processes, Math. Methods Appl. Sci. 25 (2002) 729–748.
[32] C. Walker, On diffusive and non-diffusive coalescence and breakage processes, Ph.D. Thesis, Universität Zürich, 2003.
[33] C. Walker, Asymptotic behaviour of liquid–liquid dispersions, Proc. R. Soc. Edinburgh Sect. A, to appear.
[34] D. Wilkins, A geometrical interpretation of the coagulation equation, J. Phys. A: Math. Gen. 15 (1982) 1175–1178.
[35] D. Wrzosek, Existence of solutions for the discrete coagulation–fragmentation model with diffusion, Topol. Methods Non. Anal. 9 (1997) 279–296.