STRONG SOLUTIONS TO A NONLOCAL-IN-TIME SEMILINEAR HEAT EQUATION

CHRISTOPH WALKER

ABSTRACT. Existence of strong solutions to a nonlocal semilinear heat equation is shown. The main feature of the equation is that the nonlocal term depends on the unknown on the whole time interval of existence, the latter being given a priori. The proof relies on Schauder's fixed point theorem and semigroup theory.

1. INTRODUCTION

This note is dedicated to the nonlocal problem

$$\partial_t u - \Delta u + \varphi \left(\int_0^T u(s) \, \mathrm{d}s \right) u = 0, \quad (t, x) \in (0, T] \times \Omega, \quad u|_{\partial\Omega} = 0, \quad u(0) = u^0, \tag{1.1}$$

with given potential φ , given initial datum u^0 , and given existence time T > 0, and where $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a bounded C^2 -domain. Equation (1.1) arises in the modeling of a biological nanosensor in the chaotic dynamics of a polymer chain in an aqueous solution and has been introduced and considered in [5–7]. We refer to these papers for more information on the modeling background.

Clearly, the main feature of (1.1) is that the nonlinearity depends on

$$u_T := \int_0^T u(s) \,\mathrm{d}s \,,$$

that is, on the unknown on the *whole interval of existence* [0, T], where T > 0 is *a priori* given and not free to be chosen. The problem itself is thus *not* an evolution problem in the usual sense as it does not satisfy the *Volterra property* since a solution at a time instant depends also on later times, i.e. on the future. Equations violating the Volterra property arise, of course, also in other contexts, e.g. in certain reaction-diffusion equations with non-local initial conditions [2,3], in models for long-term weather forecast (see [4] and the references therein), or in the study of stationary solutions to population models including age- and spatial structure (e.g. see [8,9]) to name but a few.

In [7] (see also [6]) the existence of weak solutions to (1.1) is shown assuming a non-negative continuous potential φ for which $s \mapsto \varphi(s)s$ is differentiable and non-decreasing. Herein we provide a simple and short proof for the existence of *strong* solutions under fairly general assumptions on the potential φ . In particular, we do not use a differentiability or monotonicity assumption. More precisely, we shall prove the following result:

Theorem 1.1. Let $\varphi \in C(\mathbb{R}, \mathbb{R})$ be non-negative and 2p > n. If either

(i)
$$u^0 \in L_\infty(\Omega)$$

or

(ii) $\varphi(s) \leq a(1+|s|), s \in \mathbb{R}$, for some a > 0 and $u^0 \in L_p(\Omega)$,

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then there is at least one strong solution

$$\in C([0,T], L_p(\Omega)) \cap C^1((0,T], L_p(\Omega)) \cap C((0,T], W_p^2(\Omega))$$

to (1.1). Moreover, $||u(t)||_p \le ||u^0||_p$ for $t \in [0,T]$. If $u^0 \ge 0$, then $u(t) \ge 0$ for $t \in [0,T]$.

The proof is an application of Schauder's fixed point theorem based on the fact that, under suitable assumptions, the operator $A(u_T) := -\Delta_D + \varphi(u_T)$ generates a semigroup $(e^{-tA(u_T)})_{t\geq 0}$ on $L_p(\Omega)$, where $-\Delta_D$ denotes the Laplacian subject to Dirichlet boundary conditions. Solutions to (1.1) are thus of the form

$$u(t) = e^{-tA(u_T)}u^0, \quad t \in [0, T],$$

and, consequently, u_T satisfies the fixed point equation

u

$$u_T = \int_0^T e^{-tA(u_T)} u^0 \,\mathrm{d}t \,. \tag{1.2}$$

As for u_T , we derive further information.

Corollary 1.2. Let u be the solution to (1.1) provided by Theorem 1.1. If $p \ge 2$, then

$$\|\nabla u_T\|_2^2 + \int_{\Omega} \varphi(u_T) |u_T|^2 \, \mathrm{d}x = \int_{\Omega} \left(u^0 - u(T) \right) u_T \, \mathrm{d}x \le 2T \|u^0\|_2^2 \, .$$

If $u^0 \in W_p^{\alpha}(\Omega)$ for some $\alpha > 0$, then $u_T \in W_p^2(\Omega)$ satisfies

$$-\Delta u_T + \varphi(u_T)u_T = u^0 - u(T) \quad in \quad \Omega, \qquad u_T = 0 \quad on \quad \partial\Omega.$$
(1.3)

In the next section we shall prove Theorem 1.1. As an immediate consequence we obtain Corollary 1.2. We shall also prove in Proposition 4.1 the uniqueness of solutions to (1.1) for small initial values or small maximal existence time provided the potential φ is Lipschitz.

The subsequent proofs equally work for similar equations, e.g. with Neumann boundary conditions. Moreover, using the fact that the heat semigroup (subject to Dirichlet boundary conditions) has exponential decay, the case $T = \infty$ can be treated the same way.

2. PROOF OF THEOREM 1.1

We let $\mathcal{L}(E, F)$ denote the space of bounded linear operators between two Banach spaces E and F with norm $\|\cdot\|_{\mathcal{L}(E,F)}$ and $\mathcal{L}(E) := \mathcal{L}(E, E)$. The norm e.g. in E is denoted by $\|\cdot\|_E$ and $\|\cdot\|_q := \|\cdot\|_{L_q(\Omega)}$ for $q \in [1, \infty]$. Given $p \in (1, \infty)$ we use the notation

$$W^{\alpha}_{p,D}(\Omega) := \begin{cases} \left\{ u \in W^{\alpha}_{p}(\Omega) \, ; \, u = 0 \text{ on } \partial \Omega \right\} & \text{ if } \quad \alpha \in \left(\frac{1}{p}, 2\right] \, , \\ \\ W^{\alpha}_{p}(\Omega) & \text{ if } \quad 0 \leq \alpha < \frac{1}{p} \, . \end{cases}$$

Moreover, for $\omega > 0$ and $\kappa \ge 1$, let $\mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega); \kappa, \omega)$ be the set of all $\mathcal{A} \in \mathcal{L}(W^2_{p,D}(\Omega), L_p(\Omega))$ such that $\omega + \mathcal{A}$ is an isomorphism from $W^2_{p,D}(\Omega)$ onto $L_p(\Omega)$ satisfying the resolvent estimates

$$\frac{1}{\kappa} \le \frac{\|(\mu + \mathcal{A})z\|_{L_p(\Omega)}}{\|\mu\| \|z\|_{L_p(\Omega)} + \|z\|_{W^2_{p,D}(\Omega)}} \le \kappa \,, \quad Re \, \mu \ge \omega \,, \quad z \in W^2_{p,D}(\Omega) \setminus \{0\}$$

Then $\mathcal{A} \in \mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega); \kappa, \omega)$ implies that $\mathcal{A} \in \mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega))$; that is, $-\mathcal{A}$ generates an analytic semigroup $(e^{-t\mathcal{A}})_{t\geq 0}$ on $L_p(\Omega)$ with domain $W^2_{p,D}(\Omega)$, see [1, I.Theorem 1.2.2]. Recall that $-\Delta_D \in \mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega))$. **Proof of Theorem 1.1, Part (i).** Assume that $\varphi \in C(\mathbb{R}, \mathbb{R})$ is non-negative and $u^0 \in L_{\infty}(\Omega)$. Since φ is then uniformly continuous and bounded on bounded sets, it follows that (considered as Nemytskii operator)

$$\varphi \in C(L_{\infty}(\Omega), L_{\infty}(\Omega))$$
 is bounded on bounded sets. (2.1)

Set $S_0 := T \|u^0\|_{\infty}$ and let

$$X_T := \bar{\mathbb{B}}_{L_\infty(\Omega)}(0, S_0)$$

denote the closed ball in $L_{\infty}(\Omega)$ of radius S_0 centered at the origin. Fix $2p \in (n, \infty)$ and note that, given any $u_T \in X_T$, the mapping $\varphi(u_T) := [w \mapsto \varphi(u_T)w] \in \mathcal{L}(L_p(\Omega))$ obviously satisfies

$$\|\varphi(u_T)\|_{\mathcal{L}(L_p(\Omega))} \le \|\varphi(u_T)\|_{\infty} \le \max_{[-S_0, S_0]} \varphi, \quad u_T \in X_T.$$

$$(2.2)$$

We now infer from the fact that $-\Delta_D \in \mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega))$ and the perturbation result [1, I.Theorem 1.3.1] that

$$A(u_T) := -\Delta_D + \varphi(u_T) \in \mathcal{H}(W^2_{p,D}(\Omega), L_p(\Omega); \kappa, \omega(S_0))$$
(2.3)

for some $\omega(S_0) > 0$ and $\kappa \ge 1$. Moreover, since φ is non-negative, we have that $-A(u_T)$ generates a positive contraction semigroup $(e^{-tA(u_T)})_{t>0}$ on each $L_q(\Omega)$ for $q \in (1, \infty]$ (which, however, is *not* strongly continuous for $q = \infty$), hence

$$|e^{-tA(u_T)}\|_{\mathcal{L}(L_q(\Omega))} \le 1, \quad t \ge 0, \quad q \in (1,\infty].$$
 (2.4)

Now, let us define

$$\Phi(u_T) := \int_0^T e^{-tA(u_T)} u^0 \,\mathrm{d}t \,, \quad u_T \in X_T \,.$$
(2.5)

Then (2.4) implies that

$$\|\Phi(u_T)\|_{\infty} \le \int_0^T \|e^{-tA(u_T)}\|_{\mathcal{L}(L_{\infty}(\Omega))} \|u^0\|_{\infty} \,\mathrm{d}t \le T \|u^0\|_{\infty} = S_0 \,, \quad u_T \in X_T \,,$$

so that $\Phi: X_T \to X_T$. Choose $2\alpha \in (n/p, 2)$ and note that (2.3) together with [1, II.Lemma 5.1.3] yield that there are $M(S_0) \ge 1$ and $\nu(S_0) > 0$ such that

$$\|e^{-tA(u_T)}\|_{\mathcal{L}(L_p(\Omega),W^{2\alpha}_{p,D}(\Omega))} \le M(S_0)e^{\nu(S_0)t}t^{-\alpha}, \quad t > 0, \quad u_T \in X_T.$$
(2.6)

Therefore,

$$\|\Phi(u_T)\|_{W^{2\alpha}_{p,D}(\Omega)} \le \int_0^T \|e^{-tA(u_T)}\|_{\mathcal{L}(L_p(\Omega), W^{2\alpha}_{p,D}(\Omega))} \|u^0\|_p \,\mathrm{d}t \le \frac{M(S_0)}{1-\alpha} e^{\nu(S_0)T} T^{1-\alpha} \|u^0\|_p \le c(S_0) \,, \tag{2.7}$$

and we conclude that $\Phi(X_T)$ is bounded in $W^{2\alpha}_{p,D}(\Omega)$, the latter being compactly embedded in $C(\overline{\Omega})$ since $2\alpha > n/p$. In order to prove the continuity of Φ let us observe that, given $u_T, v_T \in X_T$, we have

$$e^{-tA(u_T)} - e^{-tA(v_T)} = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{-(t-s)A(u_T)} e^{-sA(v_T)} \,\mathrm{d}s$$

= $-\int_0^t e^{-(t-s)A(u_T)} \left(\varphi(u_T) - \varphi(v_T)\right) e^{-sA(v_T)} \,\mathrm{d}s$, (2.8)

so that, using (2.2), (2.4), and (2.6),

$$\|e^{-tA(u_{T})} - e^{-tA(v_{T})}\|_{\mathcal{L}(L_{p}(\Omega), W^{2\alpha}_{p,D}(\Omega))}$$

$$\leq \int_{0}^{t} \|e^{-(t-s)A(u_{T})}\|_{\mathcal{L}(L_{p}(\Omega), W^{2\alpha}_{p,D}(\Omega))} \|\varphi(u_{T}) - \varphi(v_{T})\|_{\mathcal{L}(L_{p}(\Omega))} \|e^{-sA(v_{T})}\|_{\mathcal{L}(L_{p}(\Omega))} \,\mathrm{d}s \qquad (2.9)$$

$$\leq c(S_{0}) e^{c(S_{0})T} t^{1-\alpha} \|\varphi(u_{T}) - \varphi(v_{T})\|_{\infty}. \qquad (2.10)$$

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Due to the continuous embedding of $W^{2\alpha}_{p,D}(\Omega)$ in $C(\bar{\Omega})$ we derive that

$$\begin{aligned} \|\Phi(u_T) - \Phi(v_T)\|_{\infty} &\leq c \, \|\Phi(u_T) - \Phi(v_T)\|_{W^{2\alpha}_{p,D}(\Omega)} \leq c \int_0^T \|e^{-tA(u_T)} - e^{-tA(v_T)}\|_{\mathcal{L}(L_p(\Omega), W^{2\alpha}_{p,D}(\Omega))} \|u^0\|_p \, \mathrm{d}t \\ &\leq c_1(S_0) \|\varphi(u_T) - \varphi(v_T)\|_{\infty} \end{aligned}$$

for $u_T, v_T \in X_T$, hence the continuity of $\Phi : X_T \to X_T$ due to (2.1). Consequently, $\Phi \in C(X_T, X_T)$ with precompact image so that Schauder's fixed point theorem implies the existence of $u_T \in X_T$ such that $u_T = \Phi(u_T)$. We define

$$u(t) := e^{-tA(u_T)}u^0, \quad t \in [0, T],$$
(2.11)

in order to obtain a solution to (1.1). If $u^0 \ge 0$, then $u(t) \ge 0$ for $t \in [0, T]$ since the semigroup is positive. This yields part (i) of Theorem 1.1.

Proof of Theorem 1.1, Part (ii). Now assume that $\varphi \in C(\mathbb{R}, \mathbb{R})$ is non-negative and that $\varphi(s) \leq a(1 + |s|), s \in \mathbb{R}$, for some a > 0. Consider $u^0 \in L_p(\Omega)$ with 2p > n. We then adapt the prove above. Set now $S_0 := T ||u^0||_p$ and

$$X_T := \overline{\mathbb{B}}_{L_n(\Omega)}(0, S_0) \,.$$

Observe that the assumptions on φ entail

$$\varphi \in BC(X_T, L_p(\Omega)). \tag{2.12}$$

Noticing that $W_{p,D}^{2-2\varepsilon}(\Omega)$ embeds continuously in $C(\bar{\Omega})$ for $\varepsilon > 0$ small enough since 2p > n, we have, for $w \in W_{p,D}^2(\Omega)$,

$$\|\varphi(u_{T})w\|_{p} \leq c \|\varphi(u_{T})\|_{p} \|w\|_{W^{2-2\varepsilon}_{p,D}(\Omega)}$$

$$\leq c \|\varphi(u_{T})\|_{p} \|w\|_{p}^{\varepsilon} \|w\|_{W^{2}_{p,D}(\Omega)}^{1-\varepsilon}$$

$$\leq c(\delta) \|\varphi(u_{T})\|_{p}^{1/\varepsilon} \|w\|_{p} + \delta \|w\|_{W^{2}_{p,D}(\Omega)}$$
(2.13)

with $\delta > 0$ arbitrarily small. Hence, (2.12) and [1, I.Theorem 1.3.1] ensure that (2.3) holds true again. Moreover, (2.4) holds for q = p. Defining Φ as in (2.5), we argue as in part (i) to deduce that $\Phi \in C(X_T, X_T)$ has a precompact image, where the continuity follows from the fact that (2.8) along with (2.4), (2.6), and (2.13) ensure

$$\begin{aligned} \|e^{-tA(u_{T})} - e^{-tA(v_{T})}\|_{\mathcal{L}(L_{p}(\Omega))} \\ &\leq \int_{0}^{t} \|e^{-(t-s)A(u_{T})}\|_{\mathcal{L}(L_{p}(\Omega))} \|\varphi(u_{T}) - \varphi(v_{T})\|_{\mathcal{L}(W^{2-2\varepsilon}_{p,D}(\Omega),L_{p}(\Omega))} \|e^{-sA(v_{T})}\|_{\mathcal{L}(L_{p}(\Omega),W^{2-2\varepsilon}_{p,D}(\Omega))} \, \mathrm{d}s \\ &\leq c(S_{0}) \, \|\varphi(u_{T}) - \varphi(v_{T})\|_{p} \,. \end{aligned}$$

The assertion then again follows by applying Schauder's fixed point theorem.

3. PROOF OF COROLLARY 1.2

Let u be the solution to (1.1) provided by Theorem 1.1. If $p \ge 2$, then

$$\int_{s}^{T} u(\sigma) \,\mathrm{d}\sigma \in W^{2}_{2,D}(\Omega)$$

for $s \in (0,T)$ due to the regularity of the solution u. Testing (1.1) by this quantity and letting then $s \to 0^+$ yields

$$\|\nabla u_T\|_2^2 + \int_{\Omega} \varphi(u_T) |u_T|^2 \, \mathrm{d}x = \int_{\Omega} \left(u^0 - u(T) \right) u_T \, \mathrm{d}x \le \|u^0 - u(T)\|_2 \, \|u_T\|_2 \,. \tag{3.1}$$

Now, (2.4) along with (2.11) respectively (1.2) entail

$$||u(T)||_2 \le ||u^0||_2, \qquad ||u_T||_2 \le T ||u^0||_2,$$

hence

$$\|\nabla u_T\|_2^2 + \int_{\Omega} \varphi(u_T) |u_T|^2 \, \mathrm{d}x \le 2T \|u^0\|_2^2 \, .$$

If $u^0 \in W_p^{\alpha}(\Omega)$ for some $\alpha > 0$, then (1.2) implies $u_T \in W_{p,D}^2(\Omega)$ since

$$\|e^{-tA(u_T)}\|_{\mathcal{L}(W^{\alpha}_{p,D}(\Omega),W^2_{p,D}(\Omega))} \le M(S_0)e^{\nu(S_0)t}t^{\alpha/2-1}, \quad t>0.$$

due to [1, II.Lemma 5.1.3]. Integrating (1.1) with respect to $t \in (0, T)$ then gives (1.3). This proves Corollary 1.2.

4. UNIQUENESS FOR SMALL DATA

As noticed in [7] one can prove the uniqueness of solutions to (1.1) if $S_0 := T ||u^0||_{\infty}$ is small provided that, in addition,

$$\varphi : \mathbb{R} \to \mathbb{R}$$
 is locally Lipschitz continuous (4.1)

and

$$s \mapsto \varphi(s)s$$
 is non-decreasing. (4.2)

Indeed, suppose the conditions of Theorem 1.1 (i) with $p \ge 2$ and let (4.1) and (4.2) hold true. Consider two solutions u and v to (1.1) with $u(0) = v(0) = u^0 \in L_{\infty}(\Omega)$ and set $w_T := u_T - v_T$. Then

$$\|\varphi(u_T) - \varphi(v_T)\|_2 \le L(S_0) \|u_T - v_T\|_2 = L(S_0) \|w_T\|_2$$
(4.3)

for some constant $L(S_0)$ since (4.1) implies that φ is uniformly Lipschitz on the set $[-S_0, S_0]$. The same argument leading to (3.1) entails that

$$\|\nabla w_T\|_2^2 = -\int_{\Omega} \left(\varphi(u_T)u_T - \varphi(v_T)v_T\right) \left(u_T - v_T\right) dx + \int_{\Omega} \left(v(T) - u(T)\right) w_T dx$$

$$\leq \|v(T) - u(T)\|_2 \|w_T\|_2,$$
(4.4)

since (4.2) implies that the first integral is non-positive. Now, owing to (2.8), (2.4), and (4.3) we have

$$\|v(T) - u(T)\|_{2} = \| (e^{-tA(u_{T})} - e^{-tA(v_{T})}) u^{0} \|_{2}$$

$$\leq \int_{0}^{T} \| e^{-(t-s)A(u_{T})} \|_{\mathcal{L}(L_{2}(\Omega))} \|\varphi(u_{T}) - \varphi(v_{T})\|_{2} \| e^{-sA(v_{T})} \|_{\mathcal{L}(L_{\infty}(\Omega))} \|u^{0}\|_{\infty} \,\mathrm{d}s \qquad (4.5)$$

$$\leq S_{0} L(S_{0}) \|w_{T}\|_{2}.$$

Therefore, (4.4) and (4.5) entail that

$$\|\nabla w_T\|_2^2 \le S_0 L(S_0) \|w_T\|_2^2 \le c(\Omega) S_0 L(S_0) \|\nabla w_T\|_2^2,$$

where $c(\Omega)$ is the constant from Poincaré's inequality. Now, if $c(\Omega)S_0L(S_0) < 1$, then $w_T \equiv 0$ in Ω , hence $u_T = v_T$ which readily implies that the solutions u and v coincide.

However, the monotonicity condition (4.2) is not needed to prove uniqueness of solutions to (1.1) for small data as is shown in the next proposition. Moreover, in order to prove existence and uniqueness of solutions to (1.1) we also do not need a sign or growth condition on φ in this particular case and only impose the Lipschitz condition (4.1).

Proposition 4.1. Let φ satisfy (4.1), let 2p > n, and consider $u^0 \in L_p(\Omega)$. There is R > 0 such that (1.1) has a unique solution

$$u \in C([0,T], L_p(\Omega)) \cap C^1((0,T], L_p(\Omega)) \cap C((0,T], W_p^2(\Omega))$$

provided that $T ||u^0||_p \leq R$.

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Proof. We use Banach's fixed point theorem. For this fix $2\alpha \in (n/p, 2)$, let $S_0 > 0$, and put

$$X_T := \bar{\mathbb{B}}_{W^{2\alpha}_{n,D}(\Omega)}(0, S_0).$$

Note that $W_{p,D}^{2\alpha}(\Omega)$ embeds continuously into $C(\overline{\Omega})$. Thus, since φ is uniformly Lipschitz continuous on compact sets, there is a constant $L(S_0) > 0$ such that φ (considered as Nemytskii operator) satisfies

$$\|\varphi(u_T) - \varphi(v_T)\|_{\infty} \le L(S_0) \|u_T - v_T\|_{W^{2\alpha}_{p,D}(\Omega)}, \quad u_T, v_T \in X_T.$$
(4.6)

In particular, since $\|\varphi(u_T)\|_{\infty} \leq c(S_0)$, it follows as in the proof of part (i) of Theorem 1.1 that [1, I.Theorem 1.3.1] and [1, II.Lemma 5.1.3] imply (2.3) and (2.6). The latter entails, as in (2.7), that

$$\|\Phi(u_T)\|_{W^{2\alpha}_{p,D}(\Omega)} \le \frac{M(S_0)}{1-\alpha} e^{\nu(S_0)T} T^{1-\alpha} \|u^0\|_p.$$
(4.7)

Moreover, (2.10) along with (4.6) also yield

$$\|e^{-tA(u_T)} - e^{-tA(v_T)}\|_{\mathcal{L}(L_p(\Omega), W^{2\alpha}_{p, D}(\Omega))} \le c(S_0) e^{c(S_0)T} t^{1-\alpha} \|u_T - v_T\|_{W^{2\alpha}_{p, D}(\Omega)}, \quad u_T, v_T \in X_T,$$

for some $c(S_0) > 0$. Therefore

$$\begin{aligned} \|\Phi(u_T) - \Phi(v_T)\|_{W^{2\alpha}_{p,D}(\Omega)} &\leq \int_0^T \|e^{-tA(u_T)} - e^{-tA(v_T)}\|_{\mathcal{L}(L_p(\Omega), W^{2\alpha}_{p,D}(\Omega))} \|u^0\|_p \,\mathrm{d}t \\ &\leq c_1(S_0) \, e^{c(S_0)T} \, T^{2-\alpha} \, \|u^0\|_p \, \|u_T - v_T\|_{W^{2\alpha}_{p,D}(\Omega)} \,. \end{aligned}$$

Together with (4.7) this shows that $\Phi: X_T \to X_T$ is a contraction provided that $T ||u^0||_p$ is small enough. Consequently, there exists a unique $u_T \in X_T$ with $\Phi(u_T) = u_T$ if $T ||u^0||_p$ is small enough.

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LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY *Email address*: walker@ifam.uni-hannover.de