A Remark on Continuous Coagulation-Fragmentation Equations with Unbounded Diffusion Coefficients

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Abstract. Continuous coagulation-fragmentation processes with diffusion are studied. It is shown that the parameter dependent diffusion term $d(y)\Delta$ generates an analytic semigroup in suitable state spaces even for unbounded diffusion coefficients $d(y)$. This yields existence and uniqueness of local-in-time smooth solutions that are global for small initial values in the absence of fragmentation.

1. Introduction

In the present paper we demonstrate how to extend a recent result of H. Amann and the author on diffusive continuous coagulation-fragmentation equations [4] to the case of unbounded diffusion coefficients. Recall that these equations describe the time evolution of a system of a large number of particles that may change size due to coalescence or breakage. The mechanism leading to aggregation is assumed to be governed merely by Brownian motion. Applications of these processes can be found in various scientific and industrial disciplines, such as biology, physics, chemistry, or oil industry (e.g., see [9] and the references therein).

More precisely, denoting by y the particle size and by $u = u(y) = u(t, y, x)$ the particle size distribution function at time t and position x, the continuous version of the diffusive coagulation-fragmentation equations reads as

$$\begin{aligned} \partial_t u(y) - d(y)\Delta_x u(y) &= L[u](y) && \textit{in } \Omega\ , && t > 0\ , && y \in (0, y_0)\ , \\ \partial_\nu u(y) &= 0 && \textit{on } \partial\Omega\ , && t > 0\ , && y \in (0, y_0)\ , \\ u(0, y, \cdot) &= u^0(y) && \textit{in } \Omega\ , && y \in (0, y_0)\ . \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $u^0 = u^0(y, x)$ is a given initial distribution. The right hand side

$$L[u] := L_{\mathsf{b}}[u] + L_{\mathsf{c}}[u, u] + L_{\mathsf{s}}[u, u]$$

consists of the integral operators

$$L_{\mathsf{b}}[u](y) := \int_y^{y_0} \gamma(y', y)\ u(y')\ \mathrm{d}y' - u(y) \int_0^y \frac{y'}{y}\ \gamma(y, y')\ \mathrm{d}y'\ ,$$

$$
\begin{aligned}
L_{\mathsf{c}}[u,v](y) \;:=\;& \frac{1}{2}\int_0^y K(y',y-y')\;P(y',y-y')\;u(y-y')\;v(y')\;\mathrm{d}y' \\
&+\frac{1}{2}\int_y^{y_0}\int_0^{y'} K(y'',y'-y'')\;Q(y'',y'-y'') \\
&\qquad\qquad \times\;\beta_{\mathsf{c}}(y',y)\;u(y'')\;v(y'-y'')\;\mathrm{d}y''\mathrm{d}y' \\
&-u(y)\int_0^{y_0-y} K(y,y')\;\big\{P(y,y')+Q(y,y')\big\}\;v(y')\;\mathrm{d}y'\;, \\
L_{\mathsf{s}}[u,v](y) \;:=\;& \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\;\beta_{\mathsf{s}}(y',y)\;u(y'')\;v(y'-y'')\;\mathrm{d}y''\mathrm{d}y' \\
&-u(y)\int_{y_0-y}^{y_0} K(y,y')\;v(y')\;\mathrm{d}y'\;,
\end{aligned}
$$

for $y \in (0, y_0)$, where L_{b}, L_{c}, and L_{s} account for the formation and depletion of particles due to spontaneous breakage, coalescence and collisional breakage, and due to scattering, respectively. We refer to [4], [13] for a precise definition of the kernels and a more detailed interpretation of the above terms. By $Y := (0, y_0)$ we denote the admissible range for particle sizes, which is either unbounded if particles are allowed to become arbitrarily large, that is, if $y_0 = \infty$, or bounded if $y_0 \in (0,\infty)$. In the former case of the classical coagulation-fragmentation model, the scattering operator L_{s} is identically zero. As in [4] we treat both cases simultaneously.

We refrain from recalling the present state of research on continuous coagulation fragmentation processes with diffusion, but refer instead to [2], [4], [5], [8], [13] and the references therein. It has been shown in [4] that the above system of equations possesses a unique smooth solution locally in time, which preserves the total mass. Moreover, this solution exists globally for small initial values provided that the linear fragmentation terms are neglected. These results rely on the fact that the analytic semigroup generated by the size-dependent diffusion term $d(y)\Delta_x$ on the state spaces $L_1(Y, L_p(\Omega))$, $1 \le p < \infty$, has smoothing properties. The basic assumption in [4] for the generation result and the regularizing effects is that the diffusion coefficients satisfy a bound of the form

$$
0 < d_\star \le d(y) \le d^\star < \infty\;, \quad y \in Y\;.
$$

The main motivation for the present paper is to relax this restriction in order to include unbounded coefficients. We will show that neither the upper nor the lower bound for the diffusion coefficients is necessary to obtain the generation result. Nevertheless, we point out that the lower bound $d_\star > 0$ is crucial in our analysis to guarantee existence of smooth solutions. This bound provides a suitable control for the time singularities arising from the regularizing effects.

2. The Diffusion Semigroup

We briefly recall the most important notations and abbreviations already used in [4] and refer to [4] for more details. The abbreviation $L_p := L_p(\Omega)$ stands for the

Lebesgue spaces and $H^\mu_{p,\mathcal{B}} := H^\mu_{p,\mathcal{B}}(\Omega)$ for the Bessel potential spaces including Neumann boundary condition (if meaningful). For $p \in [1,\infty)$ we denote by Δ_p the (well-defined) closure in L_p of the linear operator $\Delta\,|_{C^2(\bar\Omega)}$ subject to Neumann boundary conditions, which generates a positive analytic semigroup $\{e^{t\Delta_p}\,;\,t\ge 0\}$ of contractions on L_p. The domain of Δ_p equals $H^2_{p,\mathcal{B}}$ provided $p>1$. Since the restriction of $e^{t\Delta_1}$ to L_p coincides with $e^{t\Delta_p}$, we may put $\Delta := \Delta_1$ and obtain the estimates

$$\|e^{t\Delta}\|_{\mathcal{L}(L_p,H^\alpha_{q,\mathcal{B}})} \le c\,(1\wedge t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}\ , \quad t>0\ , \tag{2.1}$$

for $\alpha\in[0,2]\setminus\{1+1/q\}$ and $1\le p\le q\le\infty$, where $q\in(1,\infty)$ if $\alpha>0$, and also

$$\|e^{t\Delta}\|_{\mathcal{L}(H^\alpha_{p,\mathcal{B}},H^\mu_{p,\mathcal{B}})} \le c\,(1\wedge t)^{-\frac{\mu-\alpha}{2}}\ , \quad t>0\ , \tag{2.2}$$

for $1<p<\infty$ and $0\le\alpha\le\mu\le 2$ with $\alpha,\mu\ne 1+1/p$.

We then set

$$\mathbb{L}_p := L_1\big(Y, L_p, (1+y)\mathrm{d}y\big)\ , \quad 1\le p\le\infty\ ,$$

and

$$\mathbb{H}^\alpha_{p,\mathcal{B}} := L_1\big(Y, H^\alpha_{p,\mathcal{B}}, (1+y)\mathrm{d}y\big)\ , \quad 1<p<\infty\ , \quad \alpha\in[0,2]\setminus\{1+1/p\}\ ,$$

with the convention $\mathbb{H}^0_{p,\mathcal{B}} := \mathbb{L}_p$ for $p\in\{1,\infty\}$. By $\mathbb{L}_p^+$ we denote the positive cone of $\mathbb{L}_p$.

Throughout this paper we assume for the diffusion coefficients that

$$d\in L_{1,loc}(Y,(0,\infty)) \tag{2.3}$$

and that there exists $d_\star$ with

$$d(y)\ge d_\star>0\ , \quad y\in Y\ . \tag{2.4}$$

If d satisfies (2.3), we define for each $p\in[1,\infty)$ an operator $\mathbb{A}_p$ by virtue of

$$(\mathbb{A}_p u)(y) := -d(y)\Delta_p u(y)\ , \quad a.e.\ y\in Y\ ,$$

for

$$u\in\mathsf{D}(\mathbb{A}_p) := \{u\in\mathbb{L}_p\,;\, u(y)\in\mathsf{D}(\Delta_p) \text{ for a.e. } y\in Y\,,\, d(\cdot)\Delta_p u\in\mathbb{L}_p\}\ .$$

Observe then that, provided d additionally satisfies (2.4), the continuous injections

$$L_1\big(Y, H^2_{p,\mathcal{B}}, d(y)(1+y)\mathrm{d}y\big) \hookrightarrow \mathsf{D}(\mathbb{A}_p) \hookrightarrow \mathbb{H}^2_{p,\mathcal{B}}\ , \quad p\in(1,\infty)\ , \tag{2.5}$$

hold. Finally, let the projection $\mathbb{P}\in\mathcal{L}(\mathbb{L}_p)$ be defined by

$$\mathbb{P}u := \frac{1}{|\Omega|}\int_\Omega u(\cdot,x)\ \mathrm{d}x\ , \quad u\in\mathbb{L}_p\ , \quad 1\le p<\infty\ ,$$

so that the space $\mathbb{L}_p$ has the direct sum decomposition

$$\mathbb{L}_p = \mathbb{P}(\mathbb{L}_p) \oplus (1-\mathbb{P})(\mathbb{L}_p)\ . \tag{2.6}$$

Note that $\mathbb{P}(\mathbb{L}_p) = L_1(Y,(1+y)\mathrm{d}y) \subset \mathbb{L}_p$. We put $\mathbb{L}_p^\bullet := (1-\mathbb{P})(\mathbb{L}_p)$ to shorten notation.

Now we can prove analogous statements to [4, Thm.2, Prop.3] for diffusion coefficients obeying (2.3) and (2.4).

Theorem 2.1. *Suppose d satisfies* (2.3) *and* (2.4)*. Then $-\mathbb{A}_p$ generates a positive strongly continuous analytic semigroup of contractions on $\mathbb{L}_p$ for each $p \in [1,\infty)$. It is given by*

$$(e^{-t\mathbb{A}_p}u)(y) = e^{td(y)\Delta_p}u(y)\ , \quad a.e.\ y \in Y\ , \quad t \ge 0\ , \quad u \in \mathbb{L}_p\ , \tag{2.7}$$

and it holds that $\mathbb{A}_p \supset \mathbb{A}_q$ for $1 \le p < q < \infty$. Furthermore, the estimates

$$\|e^{-t\mathbb{A}_p}\|_{\mathcal{L}(\mathbb{L}_p,\mathbb{H}^\alpha_{q,\mathcal{B}})} \le c(T)\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}}\ , \quad 0 < t \le T\ , \tag{2.8}$$

for $\alpha \in [0,2]\setminus\{1+1/q\}$ and $1 \le p \le q \le \infty$, where $q \in (1,\infty)$ if $\alpha > 0$, and

$$\|e^{-t\mathbb{A}_p}\|_{\mathcal{L}(\mathbb{H}^\alpha_{p,\mathcal{B}},\mathbb{H}^\mu_{p,\mathcal{B}})} \le c(T)\, t^{-\frac{\mu-\alpha}{2}}\ , \quad 0 < t \le T\ , \tag{2.9}$$

for $1 < p < \infty$ and $0 \le \alpha \le \mu \le 2$ with $\alpha, \mu \ne 1+1/p$ are valid. Moreover, for $1 \le p < \infty$, (2.6) *decomposes $e^{-t\mathbb{A}_p}$ into*

$$e^{-t\mathbb{A}_p} = 1 \oplus \left(e^{-t\mathbb{A}_p}\,|_{\mathbb{L}_p^\bullet}\right)\ , \quad t \ge 0\ ,$$

and there exists $\omega_0 > 0$ such that, for $1 < p < q \le \infty$ and some $M := M(p,q) > 0$,

$$\|e^{-t\mathbb{A}_p}\,|_{\mathbb{L}_p^\bullet}\,\|_{\mathcal{L}(\mathbb{L}_p^\bullet,\mathbb{L}_q^\bullet)} \le M\, e^{-\omega_0 t}\, t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\ , \quad t > 0\ . \tag{2.10}$$

Proof. Fix $1 \le p < \infty$. Analogously to [4, Lem.9] one shows that the tensor product $\mathcal{D}(Y)\otimes\mathcal{D}(\Omega)$ is dense in $\mathbb{L}_p$, where $\mathcal{D}(X)$ denotes the test functions on an open subset $X \subset \mathbb{R}^m$. Due to (2.3), $\mathcal{D}(Y)\otimes\mathcal{D}(\Omega)$ is contained in the domain of $\mathbb{A}_p$. Therefore, $\mathbb{A}_p$ is densely defined. That it is a closed operator in $\mathbb{L}_p$ follows from the fact that $d(y)\Delta_p$ is closed in L_p for a.e. $y \in Y$. Next we write $L_p = \mathbb{R}\cdot 1 \oplus L_p^\bullet$ with $L_p^\bullet := (1-\mathbb{P})(L_p)$ as in the proof of [4, Prop.3], which then decomposes Δ_p into $\Delta_p = 0 \oplus \Delta_p^\bullet$. Hereby, $\Delta_p^\bullet := \Delta_p\,|_{\mathsf{D}(\Delta_p)\cap L_p^\bullet}$ generates an analytic semigroup

$$\{e^{t\Delta_p^\bullet} = e^{t\Delta_p}\,|_{L_p^\bullet}\,;\, t \ge 0\}$$

on $L_p^\bullet$ and

$$[\mathsf{Re} z \ge -\omega] \subset \varrho(\Delta_p^\bullet) = \varrho(\Delta_p) \cup \{0\}$$

for some $\omega > 0$, where $\varrho(\Delta_p^\bullet)$ and $\varrho(\Delta_p)$ denote the resolvent sets of the operators $\Delta_p^\bullet$ and Δ_p, respectively. Therefore, there exist $N \ge 1$ and a number $\alpha \in (\pi/2,\pi)$ such that

$$\Sigma_\alpha := \{z \in \mathbb{C}\setminus\{0\}\,;\, |\arg z| < \alpha\} \subset \varrho(\Delta_p^\bullet)$$

and

$$\|(\lambda - \Delta_p^\bullet)^{-1}\|_{\mathcal{L}(L_p^\bullet)} \le \frac{N}{|\lambda|} , \quad \lambda \in \Sigma_\alpha .$$

Observing the resolvent decomposition

$$(\lambda - d(y)\Delta_p)^{-1} = d(y)^{-1}\left(\frac{\lambda}{d(y)} - \Delta_p\right)^{-1} = d(y)^{-1}\left[\frac{d(y)}{\lambda} \oplus \left(\frac{\lambda}{d(y)} - \Delta_p^\bullet\right)^{-1}\right]$$

for $\lambda \in \Sigma_\alpha$ and a.e. $y \in Y$, we derive

$$\|(\lambda - d(y)\Delta_p)^{-1}\|_{\mathcal{L}(L_p)} \le \frac{N'}{|\lambda|} , \quad \lambda \in \Sigma_\alpha , \quad \text{a.e. } y \in Y,$$

for some $N' \ge 1$. Clearly, this implies that Σ_α belongs to the resolvent set of the operator $-\mathbb{A}_p$ and that the resolvent is given by

$$\big((\lambda + \mathbb{A}_p)^{-1}u\big)(y) = (\lambda - d(y)\Delta_p)^{-1}u(y) , \quad a.e.\ y \in Y , \quad u \in \mathbb{L}_p , \quad \lambda \in \Sigma_\alpha .$$

Hence we deduce that

$$\|(\lambda + \mathbb{A}_p)^{-1}\|_{\mathcal{L}(\mathbb{L}_p)} \le \frac{N'}{|\lambda|} , \quad \lambda \in \Sigma_\alpha ,$$

so that well-known generation results (e.g. [6, Thm.4.2.1]) then ensure that $-\mathbb{A}_p$ is the generator of a strongly continuous analytic semigroup on $\mathbb{L}_p$ for $1 \le p < \infty$. Moreover, from [7, Thm.11.6.6] we infer (2.7). Consequently, the semigroup generated by $-\mathbb{A}_p$ is a positive semigroup of contractions and satisfies the estimates (2.8) and (2.9) in view of (2.1), (2.2) and assumption (2.4). Finally, that $e^{-t\mathbb{A}_p}$ leaves both of the the spaces $\mathbb{P}(\mathbb{L}_p)$ and $\mathbb{L}_p^\bullet$ invariant and that the estimate (2.10) holds can be shown exactly as in the proof of [4, Prop.3]. □

Remark 2.2. *Observe that* (2.4) *is required for the estimates* (2.8)-(2.10) *but not for the analyticity of the semigroup* $\{e^{-t\mathbb{A}_p} \,;\, t \ge 0\}$.

3. Well-Posedness

In the sequel, for $\vartheta \ge 0$ given, we say that *hypothesis $H(\vartheta)$ is satisfied* provided that

(H_1) K is a non-negative symmetric function defined on $Y \times Y$ and there is $k > 0$ such that

$$K(y,y')\big(d(y+y')^\vartheta + d(y)^\vartheta\big) \le k , \quad (y,y') \in Y \times Y .$$

P and Q are non-negative and symmetric functions belonging to $L_\infty(\Xi)$, where $\Xi := \{(y,y') \in Y \times Y \,;\, y+y' \in Y\}$, such that

$$0 \le P(y,y') + Q(y,y') \le 1 \quad for\ a.e.\ (y,y') \in \Xi .$$

(H_2) γ is a measurable function from $\{(y,y')\ ;\ 0<y'<y<y_0\}$ into $\mathbb{R}^+$ such that there exists $m_\gamma>0$ with

$$\int_0^y d(y')^\vartheta \gamma(y,y')\ \mathrm{d}y' + d(y)^\vartheta \int_0^y \frac{y'}{y}\gamma(y,y')\ \mathrm{d}y' \le m_\gamma \quad for\ a.e.\ y\in Y\ .$$

(H_3) β_{c} is a non-negative measurable function on $\{(y,y')\ ;\ 0<y'<y<y_0\}$ such that

$$Q(y,y')\left(\int_0^{y+y'} y''\beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' - y - y'\right) = 0 \quad for\ a.e.\ (y,y')\in\Xi\ ,$$

and there exists $m_{\mathsf{c}}>0$ with

$$Q(y,y')\int_0^{y+y'} d(y'')^\vartheta \beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' \le m_{\mathsf{c}} \quad for\ a.e.\ (y,y')\in\Xi\ .$$

(H_4) β_{s} is a measurable function from $(y_0,2y_0)\times(0,y_0)$ into $\mathbb{R}^+$ such that

$$\int_0^{y_0} y''\beta_{\mathsf{s}}(y+y',y'')\ \mathrm{d}y'' = y+y' \quad for\ a.e.\ y+y'\in(y_0,2y_0)\ ,$$

and there exists $m_{\mathsf{s}}>0$ with

$$\int_0^{y_0} d(y'')^\vartheta \beta_{\mathsf{s}}(y+y',y'')\ \mathrm{d}y'' \le m_{\mathsf{s}} \quad for\ a.e.\ y+y'\in(y_0,2y_0)\ .$$

Clearly, hypothesis $H(\vartheta)$ with $\vartheta\ge 0$ implies hypothesis $H(0)$ in view of (2.4). Let us mention right now that $H(\vartheta)$ ensures 'the gain of integrability' (see [12, Lem.2.6])

$$L[u]\in L_1\big(Y,d(y)^\vartheta(1+y)\mathrm{d}y\big) \quad \text{for} \quad u\in L_1(Y,(1+y)\mathrm{d}y)\ . \tag{3.1}$$

Also note that hypothesis $H(0)$ coincides with the assumptions made in [4]. We therefore refer to [4, Ex.1] for kernels obeying $H(0)$. If, in the case $y_0<\infty$, the kernels are of the form

$$\begin{aligned}
&\gamma(y,y') \propto y^{\alpha-\xi-1}\,(y')^\xi\ , && 0<y'<y<y_0\ ,\\
&\beta_{\mathsf{c}}(y,y') := (\zeta+2)\,y^{-1-\zeta}\,(y')^\zeta\ , && 0<y'<y<y_0\ ,\\
&\beta_{\mathsf{s}}(y,y') := (\nu+2)\,y_0^{-2-\nu}\,y\,(y')^\nu\ , && 0<y'<y_0\le y<2y_0\ ,\\
&Q(y,y') \propto (y+y')^\tau\ , && (y,y')\in\Xi\ ,\\
&K(y,y') \propto (yy')^\mu\ , && y,y'\in Y\ ,
\end{aligned}$$

with $0\ge\xi,\zeta,\nu>-1$ and $\alpha,\mu,\tau>0$, we can choose $\vartheta>0$ sufficiently small such that hypothesis $H(\vartheta)$ is satisfied for diffusion coefficients $d(y)\sim y^{-\lambda}$, $\lambda>0$.

In order to show that problem (1.1) is well-posed, we assume in the sequel that (2.3), (2.4), and at least hypothesis $H(0)$ are satisfied. Then it follows, for $1\le p<\infty$, that

$$L_{\mathsf{b}}\in\mathcal{L}(\mathbb{L}_p) \quad and \quad G := L_{\mathsf{c}} + L_{\mathsf{s}} \in \mathcal{L}^2(\mathbb{L}_{2p},\mathbb{L}_p) \tag{3.2}$$

since the pointwise product $L_{2p} \times L_{2p} \to L_p$ is a multiplication. If J denotes an interval in $\mathbb{R}^+$ containing 0, we put $\dot{J} := J \setminus \{0\}$. We then call $u \in C(J, \mathbb{L}_p)$ a *mild* $\mathbb{L}_p$-solutions to the re-written problem

$$\dot{u} + \mathbb{A}_p u = L[u] \ , \quad t > 0 \ , \quad u(0) = u^0 \ , \tag{3.3}$$

provided it solves the fixed point equation

$$u(t) = U(t)u^0 + U \star L[u](t) \quad \text{in} \quad \mathbb{L}_p \ , \quad t \in J \ , \tag{3.4}$$

where $U(t) := e^{-t\mathbb{A}}$, $t \geq 0$, with $\mathbb{A} := \mathbb{A}_1$ and

$$U \star v(t) := \int_0^t U(t-s)\, v(s)\, \mathrm{d}s \ , \quad t \in J \ .$$

A mild solution is a *strong* $\mathbb{L}_p$-solution if $u \in C^1(\dot{J}, \mathbb{L}_p) \cap C(\dot{J}, \mathsf{D}(\mathbb{A}_p))$.
Given a Banach space E and $\mu \in \mathbb{R}$, we denote by $BC_\mu(\dot{J}, E)$ the Banach space of all functions $u : \dot{J} \to E$ such that $\big(t \mapsto t^\mu u(t)\big)$ is bounded and continuous from $\dot{J}$ into E, equipped with the norm

$$u \mapsto \|u\|_{BC_\mu(\dot{J},E)} := \sup_{t \in \dot{J}} t^\mu \, \|u(t)\|_E \ .$$

We write $C_\mu(\dot{J}, E)$ for the closed linear subspace thereof consisting of all u satisfying $t^\mu u(t) \to 0$ in E as $t \to 0$.

Due to Theorem 2.1 we have (see [4, Prop.4])

Proposition 3.1. *Let $1 \leq p \leq q \leq \infty$ and $\alpha \in [0,2) \setminus \{1 + 1/q\}$ be such that $n(1/p - 1/q)/2 + \alpha/2 < 1$ and either $q \in (1, \infty)$ or $\alpha = 0$. Then, for $\mu < 1$,*

$$\big(u \mapsto U \star L_{\mathsf{b}}[u]\big) \in \mathcal{L}\big(C_\mu(\dot{J}, \mathbb{L}_p), C_{\mu + \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\alpha}{2} - 1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\big) \tag{3.5}$$

and

$$\big(u \mapsto U \star G[u,u]\big) \in \mathcal{L}^2\big(C_{\mu/2}(\dot{J}, \mathbb{L}_{2p}), C_{\mu + \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\alpha}{2} - 1}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})\big) \ . \tag{3.6}$$

Moreover, the analogue of [4, Prop.6]) is still valid:

Proposition 3.2. *Let $1 < p \leq q < \infty$, $\alpha \in [0,2) \setminus \{1 + 1/q\}$ and assume that $n(1/p - 1/q)/2 + \alpha/2 < 1$, where either $\alpha = 0$ and $p < q$ or $\alpha > 0$ and $q \in (1, \infty)$. Then, for $u^0 \in \mathbb{L}_p$,*

$$Uu^0 := \big(t \mapsto U(t)u^0\big) \in C_{\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{\alpha}{2}}(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}}) \ .$$

Proof. Owing to (2.5), $n(1/p - 1/q)/2 + \alpha/2 < 1$, and classical embedding arguments we have

$$\mathsf{D}(\mathbb{A}_p) \hookrightarrow \mathbb{H}^2_{p,\mathcal{B}} \hookrightarrow \mathbb{H}^\alpha_{q,\mathcal{B}} \ .$$

Hence, since $U(t)u^0 \in \mathsf{D}(\mathbb{A}_p)$, $t > 0$, due to Theorem 2.1, we deduce

$$\|U(t+h)u^0 - U(t)u^0\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}} \leq c \int_Y \|e^{(t+h)d(y)\Delta} u^0(y) - e^{td(y)\Delta} u^0(y)\|_{H^2_{p,\mathcal{B}}} \, (1+y)\, \mathrm{d}y$$

for $0 \le |h| < t$. Taking (2.1) and (2.4) into account and that $\{e^{\tau\Delta}\,;\,\tau \ge 0\}$ is a bounded and strongly continuous semigroup on $H^2_{p,\mathcal{B}}$ in view of [1, V.Thm.2.1.3], we infer from Lebesgue's theorem that the right hand side of the above inequality tends to zero as $h \to 0$, and thus $Uu^0 \in C(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})$. Theorem 2.1 then yields that the map $t \mapsto t^\zeta \|U(t)u^0\|_{\mathbb{H}^\alpha_{q,\mathcal{B}}}$ remains bounded on $\dot{J}$ for $\zeta := n(1/p - 1/q)/2 + \alpha/2 < 1$. The fact that $\mathbb{H}^\alpha_{q,\mathcal{B}}$ is dense in $\mathbb{L}_p$ implies $Uu^0 \in C_\zeta(\dot{J}, \mathbb{H}^\alpha_{q,\mathcal{B}})$ as in the proof of [4, Prop.6]. □

Under hypothesis $H(0)$ we can prove now existence and uniqueness of mild solutions to problem (3.3). Subsequently, we will derive more regularity (with respect to time) for these solutions.
Our global-in-time statement requires that collisional breakage is dominated by coalescence, i.e., that

$$Q(y,y')\left(\int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'')\,\mathrm{d}y'' - 2\right) \le P(y,y')\ , \quad y+y' \in Y\ , \tag{3.7}$$

and that scattering is a binary processes, meaning that

$$\beta_{\mathsf{s}}(y,y') = \beta_{\mathsf{s}}(y,y-y')\ , \quad (y,y') \in \Xi\ , \tag{3.8}$$

and

$$\beta_{\mathsf{s}}(y,y') = 0\ , \quad 0 < y' < y - y_0 < y_0\ . \tag{3.9}$$

Then we have the following existence and uniqueness result.

Theorem 3.3. *Let* (2.3), (2.4), *and hypothesis* $H(0)$ *be satisfied and assume that* $(n/2 \vee 1) < p < \infty$. *Then, given any non-negative initial value* $u^0 \in \mathbb{L}_p$, *problem* (3.3) *possesses a unique maximal non-negative mild* $\mathbb{L}_p$*-solution* $u := u(\cdot; u^0)$ *on an open interval* $J(u^0) \subset \mathbb{R}^+$ *such that*

$$t^{n/4p}\,\|u(t)\|_{\mathbb{L}_{2p}} \to 0 \quad as \quad t \to 0^+\ .$$

In addition, u *has the regularity*

$$u \in C(\dot{J}(u^0), \mathbb{H}^2_{q,\mathcal{B}})\ , \quad q \in (1,\infty)\ . \tag{3.10}$$

If $t^+ := \sup\, J(u^0) < \infty$, *then*

$$\sup_{t^+/2 < t < t^+} \|u(t)\|_{\mathbb{L}_q} = \infty\ , \quad (n/2 \vee 1) < q < \infty\ . \tag{3.11}$$

The solution u *preserves the total mass, that is,*

$$\int_\Omega \int_Y y\,u(t,y,x)\,\mathrm{d}y\mathrm{d}x = \int_\Omega \int_Y y\,u^0(y,x)\,\mathrm{d}y\mathrm{d}x\ , \quad t \in J(u^0)\ . \tag{3.12}$$

Finally, if $\gamma \equiv 0$ *and (3.7)-(3.9) are satisfied, then, for* $p \in (n/2,\infty)$ *and* $p \ge 2$, *the solution remains bounded in* $\mathbb{L}_p$ *whenever* $\|u^0\|_{\mathbb{L}_p}$ *is small. In particular,* u *exists globally in time in this case.*

Proof. The proof of the existence and uniqueness parts of this theorem including the blow-up behavior (3.11) are, verbatim, the same as the one of [4, Thm.7] (see step (i) and (iii) of the proof of [4, Thm.7] and Propositions 3.1 and 3.2). For the regularity result (3.10) we may focus on $q > p$. We choose $T \in \dot{J}(u^0)$ and $\varepsilon > 0$ sufficiently small and derive,

$$L[u(\cdot+\varepsilon)] \in C([0,T], \mathbb{H}^{\nu}_{q,\mathcal{B}}) \quad \text{and} \quad u \in C((0,T], \mathbb{H}^{\nu}_{q,\mathcal{B}})$$

for some $\nu > 0$. This can be done as in the second step of the proof of [4, Thm.7] in view of Propositions 3.1 and 3.2. Since U is strongly continuous on $\mathbb{H}^{\nu}_{q,\mathcal{B}}$ due to Lebesgue's theorem (see the proof of Proposition 3.2), estimate (2.9) implies

$$U \star L[u(\cdot+\varepsilon)] \in C([0,T], \mathbb{H}^{2}_{q,\mathcal{B}}) \quad \text{and} \quad Uu(\varepsilon) \in C((0,T], \mathbb{H}^{2}_{q,\mathcal{B}}) \ .$$

Due to

$$u(t+\varepsilon) = U(t)u(\varepsilon) + U \star L[u(\cdot+\varepsilon)](t) \ , \quad t \in [0,T] \ , \tag{3.13}$$

assertion (3.10) follows since $\varepsilon > 0$ was arbitrary.
Next observe that, for $T \in \dot{J}(u^0)$, the mild solution $u = u(\cdot; u^0)$ is bounded on $[0,T]$ with values in $\mathbb{H}^{\alpha}_{q,\mathcal{B}}$ provided $u^0 \in \mathbb{H}^{\alpha}_{q,\mathcal{B}}$. Since $\mathbb{H}^{\alpha}_{q,\mathcal{B}}$ embedds continuously in $\mathbb{L}_\infty$ for q sufficiently large and since u solves the fixed point equation

$$u(t) = U_\omega(t)u^0 + U_\omega \star (L[u] + \omega u)(t) \ , \quad t \in J(u^0) \ ,$$

we deduce that u is non-negative for non-negative initial values $u^0 \in \mathbb{H}^{\alpha}_{q,\mathcal{B}}$ as in [4, Thm.10]. Here, the constant $\omega > 0$ suitably depends on $\sup_{0\le t\le T} \|u(t)\|_{\mathbb{L}_\infty}$ and U_ω is given by $U_\omega(t) := e^{-\omega t}U(t)$. The continuous dependence of $u(\cdot; u^0)$ on u^0 and the density of $\mathbb{H}^{\alpha}_{q,\mathcal{B}} \cap \mathbb{L}_p^+$ in $\mathbb{L}_p^+$ (see [4, Lem.9]) yields then the positivity assertion.
The conservation of mass formula (3.12) follows from (3.4) by taking into account the equality

$$\begin{aligned}\int_Y y \int_\Omega [U(t)v](y)\ \mathrm{d}x\,\mathrm{d}y &= \int_Y y \int_\Omega e^{td(y)\Delta} v(x,y)\ \mathrm{d}x\,\mathrm{d}y \\ &= \int_Y y \int_\Omega v(x,y)\ \mathrm{d}x\,\mathrm{d}y \ , \quad t > 0 \ , \quad v \in \mathbb{L}_p \ ,\end{aligned}$$

which is due to the Neumann boundary conditions, and observing that

$$\int_Y y\, L[v](y)\ \mathrm{d}y \ = \ 0 \ , \quad v \in L_1(Y, y\mathrm{d}y) \ ,$$

(see hypotheses (H_1)-(H_4) and [12, Lem.2.6]). Finally, based on estimate (2.10), the global-in-time existence result is obtained as in [4, Thm.16] noticing that the proof of the latter merely requires mild solutions. This proves the assertion. $\square$

Before we prove that hypothesis $H(\vartheta)$ with $\vartheta > 0$ provides strong solutions, let us show that also the weaker assumption $H(0)$ implies that the original equation (1.1) is satisfied pointwise almost everywhere. More precisely, defining

$$\mathfrak{L}_q := L_1(Y, L_q, \mathrm{d}\mu(y)) \quad \text{with} \quad \mathrm{d}\mu(y) := d(y)^{-1}(1+y)\mathrm{d}y \ ,$$

we have

Theorem 3.4. *The mild solution $u = u(\cdot; u^0)$ provided by Theorem 3.3 belongs to $C^1(\dot{J}(u^0), \mathfrak{L}_q)$ for each $q \in [1, \infty)$ and it holds that*

$$\dot{u}(t) = d\Delta u(t) + L[u(t)] \quad in \quad \mathfrak{L}_q \ , \quad t \in \dot{J}(u^0) \ .$$

In particular, for almost every $y \in Y$, it holds that

$$\dot{u}(t, y) - d(y)\Delta u(t, y) = L[u(t)](y) \quad in \quad L_q \ , \quad t \in \dot{J}(u^0) \ .$$

Proof. Fix $T \in \dot{J}(u^0)$ and $\varepsilon > 0$ sufficiently small and set

$$u_\varepsilon := u(\cdot + \varepsilon) \in C([0, T], \mathbb{H}^2_{q,\mathcal{B}}) \ , \quad q \in (1, \infty) \ .$$

Then there is $\nu > 0$ such that $f_\varepsilon := L[u_\varepsilon] \in C([0, T], \mathbb{H}^\nu_{q,\mathcal{B}})$ (see [10, Cor.4.5.2]). Given $t \in [0, T]$ and $h > 0$ sufficiently small, we deduce

$$\begin{aligned}
&\left\| h^{-1}\big(U(t + h)u(\varepsilon) - U(t)u(\varepsilon)\big) + \mathbb{A}\, U(t)u(\varepsilon) \right\|_{\mathfrak{L}_q} \\
&\qquad \le h^{-1} \int_Y d(y) \int_t^{t+h} \left\| \Delta \left(e^{\tau d(y)\Delta} - e^{t d(y)\Delta}\right) u_\varepsilon(y) \right\|_{L_q} \mathrm{d}\tau \, \mathrm{d}\mu(y) \\
&\qquad \le h^{-1} \int_Y \int_t^{t+h} \left\| \left(e^{\tau d(y)\Delta} - e^{t d(y)\Delta}\right) u_\varepsilon(y) \right\|_{H^2_{q,\mathcal{B}}} \mathrm{d}\tau \, (1 + y)\mathrm{d}y \ .
\end{aligned}$$

Since $\{e^{\tau d(y)\Delta} \, ; \, \tau \ge 0\}$ is for a.e. $y \in Y$ a bounded and strongly continuous semigroup on $H^2_{q,\mathcal{B}}$, we may apply Lebesgue's theorem to deduce that the right hand side of the above inequality tends to zero as h does. Hence

$$\partial_+ U(t)u(\varepsilon) = -\mathbb{A}\, U(t)u(\varepsilon) \quad \text{in} \quad \mathfrak{L}_q \ , \quad t \in [0, T] \ , \tag{3.14}$$

and, since by the same arguments $\partial_+ U u(\varepsilon) \in C([0, T], \mathfrak{L}_q)$, we obtain that $Uu(\varepsilon)$ belongs to $C^1([0, T], \mathfrak{L}_q)$ with derivative given by (3.14). Next, for $h > 0$ small and $t \in (0, T]$, we write

$$\begin{aligned}
h^{-1}\big(U \star f_\varepsilon(t + h) - U \star f_\varepsilon(t)\big) =\ & h^{-1} \int_0^t \big(U(t + h - s) - U(t - s)\big) f_\varepsilon(s) \, \mathrm{d}s \\
& + h^{-1} \int_t^{t+h} U(t + h - s) f_\varepsilon(s) \, \mathrm{d}s \\
=:\ & M_h + N_h \ .
\end{aligned}$$

Then, as above, we compute

$$\begin{aligned}
&\| M_h + \mathbb{A}\,(U \star f_\varepsilon)(t) \|_{\mathfrak{L}_q} \\
&\qquad \le c h^{-1} \int_Y \int_0^t \int_t^{t+h} \left\| \left(e^{(\tau - s)d(y)\Delta} - e^{(t-s)d(y)\Delta}\right) f_\varepsilon(s, y) \right\|_{H^2_{q,\mathcal{B}}} \mathrm{d}\tau \, \mathrm{d}s \, (1 + y)\mathrm{d}y \ .
\end{aligned}$$

According to (2.2) and (2.4), for $h \le 1$ and $0 < s < t \le T$, it holds that

$$\begin{aligned}
h^{-1} \int_t^{t+h} & \left\| \left(e^{(\tau - s)d(y)\Delta} - e^{(t-s)d(y)\Delta}\right) f_\varepsilon(s, y) \right\|_{H^2_{q,\mathcal{B}}} \mathrm{d}\tau \\
& \le c \sup_{t \le \tau \le t+1} \big(1 \wedge (\tau - s)d(y)\big)^{-1+\nu/2} \, \| f_\varepsilon(s, y) \|_{H^\nu_{q,\mathcal{B}}} \\
& \le c(T)\, (t - s)^{-1+\nu/2} \, \| f_\varepsilon(s, y) \|_{H^\nu_{q,\mathcal{B}}}
\end{aligned}$$

so that we may again apply Lebesgue's theorem to deduce

$$M_h \longrightarrow -\mathbb{A}\,(U\star f_\varepsilon)(t) \quad \text{in} \quad \mathfrak{L}_q\ , \quad h\to 0^+\ . \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}\|N_h - f_\varepsilon(t)\|_{\mathfrak{L}_q} \le h^{-1}\int_Y\int_t^{t+h} \|e^{(t+h-s)d(y)\Delta}\|_{\mathcal{L}(L_q)}\|f_\varepsilon(s,y)-f_\varepsilon(t,y)\|_{L_q}\,\mathrm{d}s\,\mathrm{d}\mu(y)\\ + h^{-1}\int_Y\int_t^{t+h}\big\|e^{(t+h-s)d(y)\Delta}f_\varepsilon(t,y)-f_\varepsilon(t,y)\big\|_{L_q}\,\mathrm{d}s\,\mathrm{d}\mu(y)\\ \le d_\star^{-1}\sup_{t\le s\le t+h}\|f_\varepsilon(s)-f_\varepsilon(t)\|_{\mathbb{L}_q}\\ + d_\star^{-1}\int_Y \sup_{0<\tau<hd(y)}\|e^{\tau\Delta}f_\varepsilon(t,y)-f_\varepsilon(t,y)\big\|_{L_q}\,(1+y)\,\mathrm{d}y\ .\end{aligned}$$

Lebesgue's theorem shows that the right side converges towards zero as $h\to 0^+$, whence $N_h\to f_\varepsilon(t)$ in $\mathfrak{L}_q$. Taking (3.15) into account we obtain

$$\partial_+(U\star f_\varepsilon)(t) = -\mathbb{A}\,U\star f_\varepsilon(t) + f_\varepsilon(t) \quad \text{in} \quad \mathfrak{L}_q\ , \quad t\in(0,T]\ . \tag{3.16}$$

Due to (2.2), (2.4), and the strong continuity of U on $\mathbb{H}^\nu_{q,\mathcal{B}}$, we derive

$$\partial_+(U\star f_\varepsilon)\in C((0,T],\mathfrak{L}_q) \quad \text{and} \quad U\star f_\varepsilon\in C((0,T],\mathfrak{L}_q)\ .$$

Thus $U\star f_\varepsilon\in C^1((0,T],\mathfrak{L}_q)$. Recalling (3.14), we infer from equation (3.13) that $u_\varepsilon\in C^1((0,T],\mathfrak{L}_q)$. Since $\varepsilon>0$ was arbitrary, the assertion follows. □

Stronger assumptions on the kernels ensure that the mild solutions actually are strong solutions as the concluding theorem shows.

Theorem 3.5. *Suppose that $H(\vartheta)$ holds with $\vartheta>0$. Then, given the assumptions of Theorem 3.3, the mild solution $u=u(\cdot;u^0)$ is a strong $\mathbb{L}_p$-solution to problem* (3.3) *and has the additional regularity*

$$u\in C^1(\dot J(u^0),\mathbb{L}_q)\cap C(\dot J(u^0),\mathsf{D}(\mathbb{A}_q))\ , \quad q\in(1,\infty)\ .$$

Proof. We may assume that $q\in(1,\infty)$ is large. By $[\cdot,\cdot]_\theta$ we denote the complex interpolation functor of exponent $\theta\in(0,1)$. Then, in view of [11, Thm.1.18.4], we may follow the lines of the second step of the proof of [11, Thm.1.18.5] to show that

$$\big[L_1(Y,L_q,(1+y)\mathrm{d}y), L_1(Y,H^2_{q,\mathcal{B}},d(y)(1+y)\mathrm{d}y)\big]_\theta = L_1\big(Y,H^{2\theta}_{q,\mathcal{B}},d(y)^\theta(1+y)\mathrm{d}y\big)$$

provided $2\theta\in(0,1)\setminus\{1+1/q\}$. Consequently, (2.5) implies, for $2\theta\ne 1+1/q$,

$$L_1\big(Y,H^{2\theta}_{q,\mathcal{B}},d(y)^\theta(1+y)\mathrm{d}y\big)\hookrightarrow\big[\mathbb{L}_q,\mathsf{D}(\mathbb{A}_q)\big]_\theta =: \mathbb{E}_\theta\ . \tag{3.17}$$

Let now $u\in C(J(u^0),\mathbb{L}_p)\cap C(\dot J(u^0),\mathbb{H}^2_{q,\mathcal{B}})$ be the mild solution provided by Theorem 3.3. Since q is large, we infer from [10, Cor.4.5.2] that there is some $\nu>0$ small such that the pointwise product $H^2_{q,\mathcal{B}}\times H^2_{q,\mathcal{B}}\hookrightarrow H^{2\nu}_{q,\mathcal{B}}$ is continuous. By making ϑ smaller if necessary, we may assume that $\nu=\vartheta$. Therefore, hypothesis

$H(\vartheta)$ guarantees that the map $t \mapsto L[u(t)]$ is continuous on $\dot{J}(u^0)$ with values in $L_1(Y, H^{2\vartheta}_{q,\mathcal{B}}, d(y)^\vartheta(1+y)\mathrm{d}y)$ (see (3.1)). Taking (3.17) into account we thus obtain $L[u] \in C(\dot{J}(u^0), \mathbb{E}_\vartheta)$. As in part (ii) of the proof of [4, Thm.7] we can now shift the equation (3.3) and apply [1, Thm.IV.1.5.1] in order to deduce that $u(\cdot + \varepsilon)$ is a strong $\mathbb{L}_q$-solution for each $\varepsilon > 0$ small. This proves the claim. $\square$

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