# POSITIVE EQUILIBRIUM SOLUTIONS FOR AGE AND SPATIALLY STRUCTURED POPULATION MODELS

### CHRISTOPH WALKER

ABSTRACT. The existence of positive equilibrium solutions to age-dependent population equations with nonlinear diffusion is studied in an abstract setting. By introducing a bifurcation parameter measuring the intensity of the fertility it is shown that a branch of (positive) equilibria bifurcates from the trivial equilibrium. In some cases the direction of bifurcation is analyzed.

### 1. INTRODUCTION

The present paper is dedicated to the study of nontrivial equilibrium (i.e. nonzero time-independent) solutions to abstract age-structured population models with nonlinear diffusion, that is, to equations of the form

$$\partial_t u + \partial_a u + A(u, a) u + \mu(u, a) u = 0, \qquad t > 0, \quad a \in (0, a_m), \qquad (1.1)$$

$$u(t,0) = \int_0^{a_m} \beta(u,a) \, u(a) \, \mathrm{d}a \,, \qquad \qquad t > 0 \,, \qquad (1.2)$$

subject to some initial condition at t = 0. Here, u = u(t, a) is a function taking on values in some Banach space  $E_0$  and represents in applications the density at time t of a population of individuals structured by age  $a \in J := [0, a_m)$ , where  $a_m \in (0, \infty]$  is the maximal age. The real-valued functions  $\mu = \mu(u, a)$ and  $\beta = \beta(u, a)$  are respectively the death and birth modulus. The operator A(u, a) depending in a certain way on the density u specified later governs the spatial movement of individuals. It is assumed to be a (unbounded) linear operator  $A(u, a) : E_1 \subset E_0 \to E_0$  satisfying additional technical assumptions given later.

Age-structured models have a long history and various aspects regarding well-posedness and behavior for large times were investigated (see [33] and the references therein) though most research was devoted to models neglecting spatial structure from the outset or considering merely linear diffusion, see e.g. [14, 16, 5, 19, 23, 27] and the references therein. Less seems to be known for the case of age-structured models with nonlinear diffusion (however, see e.g. [4, 17, 29, 30, 18]).

The understanding of the large time behavior of age-structured populations whose evolution is governed by equations (1.1), (1.2) requires in particular precise information about the existence of equilibrium solutions. Since obviously  $u \equiv 0$  is such an equilibrium solution the aim is to establish existence of nontrivial equilibria. Moreover, since u represents a density the main task is to single out the positive equilibrium solutions in the (ordered) space  $E_0$ . The aim of this paper is to give an existence result under "natural" assumptions on A,  $\mu$ , and  $\beta$  and this will be done in the framework of bifurcation theory.

Equilibria of (1.1), (1.2) are solutions to

$$\partial_a u + A(u, a) u + \mu(u, a) u = 0, \qquad a \in (0, a_m),$$
(1.3)

$$u(0) = \int_0^{a_m} \beta(u, a) \, u(a) \, \mathrm{d}a \,. \tag{1.4}$$

Key words and phrases. Age structure, nonlinear diffusion, population models, bifurcation. *Mathematics Subject Classifications* (2000): 35K55, 35K90, 92D25.

Suppose that for any fixed u the map

$$a \mapsto \mathbb{A}(u, a) := A(u, a) + \mu(u, a)$$

generates a parabolic evolution operator  $\Pi_u(a, \sigma)$ ,  $0 \le \sigma \le a < a_m$ , on  $E_0$  (e.g. in the sense of [2]). In particular,  $\Pi_u(a, 0)$  is (for u and a fixed) a bounded linear operator on  $E_0$  such that  $v(a) := \Pi_u(a, 0)v^0$  for  $v^0 \in E_0$  is the unique strong solution to

$$\partial_a v + \mathbb{A}(u, a)v = 0$$
,  $a \in (0, a_m)$ ,  $v(0) = v^0$ .

Thus, any solution u to (1.3) is necessarily of the form

$$u(a) = \Pi_u(a, 0)u(0) , \quad a \in J .$$
(1.5)

Substituting this into (1.4) one derives the relation

$$u(0) = Q(u)u(0) , (1.6)$$

where the linear operator Q(u) on  $E_0$  (for u fixed) is defined by

$$Q(u) := \int_0^{a_m} \beta(u, a) \Pi_u(a, 0) \,\mathrm{d}a \,. \tag{1.7}$$

Therefore, u is a solution to (1.3), (1.4) if and only if it satisfies (1.5) and (1.6). Consequently, u(0) is (if nonzero) an eigenvector corresponding to the eigenvalue 1 of the linear operator Q(u) on  $E_0$ .

Roughly speaking, Q(u) contains information about the spatial distribution of the expected number of newborns that an individual produces over its lifetime when the population's distribution is u. In the present paper we suggest a bifurcation problem by introducing a bifurcation parameter n which determines the intensity of the fertility without changing its structure. More precisely, we are interested in nontrivial solutions (n, u) (that is,  $u \neq 0$ ) to

$$\partial_a u + A(u, a) u + \mu(u, a) u = 0, \qquad a \in (0, a_m),$$
(1.8)

$$u(0) = n \int_{0}^{a_{m}} b(u, a) u(a) \,\mathrm{d}a \,, \tag{1.9}$$

where we put

$$n b(u, a) := \beta(u, a), \qquad (1.10)$$

with b being a normalized function such that the spectral radius of the bounded linear operator

$$Q_0 := \int_0^{a_m} b(0, a) \Pi_0(a, 0) \,\mathrm{d}a$$

equals 1, that is,

$$r(Q_0) = 1. (1.11)$$

Note that under this normalization we have  $r(Q(0)) = nr(Q_0) = n$ ; the bifurcation parameter n is thus the spectral radius of the "inherent spatial net production rate at low densities" (technically when  $u \equiv 0$ ). If  $r(Q_0)$  is an eigenvalue of  $Q_0$ , then (1.11) may be interpreted as that there exists a distribution for which the population is at balance meaning that the birth processes yield exact replacement (provided that death and birth modulus and spatial displacement are described by  $\mu(0, \cdot), \beta(0, \cdot)$ , and  $A(0, \cdot)$ ).

In this paper we provide a set of *n*-values for which (1.8), (1.9) have nontrivial and positive solutions around the critical value n = 1 and  $u \equiv 0$ , analogously to the "spatially homogeneous" case (i.e. when A = 0), see [8]. More precisely, it is shown that a branch of nontrivial solutions bifurcates from (i.e. intersects with) the branch of trivial solutions  $(n, u) = (n, 0), n \in \mathbb{R}$ , at the critical value of *n*. In principle, the direction at which bifurcation occurs will be related to (the values at  $u \equiv 0$  of the derivatives of)  $\mu$ ,  $\beta$ , and A by computing a parametrization of the branch of nontrivial solutions. In some cases, the direction can be computed explicitly. In particular, examples will be given where supercritical bifurcation occurs. In view of the arguments given in [8] for the spatially homogeneous version of (1.8), (1.9) indicating a loss of stability of the trivial solution when the parameter n increases through the critical value, the same is expected for the present situation with nonlinear diffusion though not proved herein.

The bifurcation result of this paper has a local character. However, we refer to a forthcoming paper [31] where global bifurcation (i.e. the existence of an unbounded continuum of solutions (n, u)) will be shown for the case of age-structured equations with diffusion possibly containing nonlinearities in "lower order terms".

In order to derive the local bifurcation result we consider problem (1.8), (1.9) in a more general framework so that the results actually apply to a broader range of similar problems. In Section 2 we investigate the general abstract framework and prove the bifurcation result using findings based on the implicit function theorem obtained in [7]. We also derive a more precise characterization of the nontrivial branch of solutions and show that the equilibria are positive. The subsequent Section 3 then gives applications for these results (see also Subsection 1 below). We shall point out that analogue results for populations structured by age only (that is, when A = 0) were derived [8, 9, 10]. Furthermore, additional results regarding equilibrium solutions for age-structured equations are to be found in e.g. [12, 13, 15, 16, 20, 21, 22, 32, 33] and the references therein.

Clearly, the abstract approach chosen in this paper applies to many different situation. To explain our approach in more detail and outline the functional setting in concrete applications, we conclude the introduction with an informal discussion of a simple example. We postpone all technical details to Section 3 (see, in particular, Example 3.4), where the precise assumptions needed will be stated.

An Example. Let u = u(a, x) denote the distribution density of individuals of a population with age  $a \in (0, a_m)$  at spatial position x in a bounded space region  $\Omega$ , where  $a_m \in (0, \infty)$  denotes the maximal age. Suppose that the individuals move within  $\Omega$  and that dispersal speed a > 0 depends smoothly on the local overall population; that is, suppose that movement is governed by a density dependent diffusion term  $-\operatorname{div}_x(a(U(x))\nabla_x u)$ , where  $U(x) := \int_0^{a_m} u(a, x) da$  is the overall population at spatial position  $x \in \Omega$ . Assume further that individuals cannot leave the space region  $\Omega$  so that the behavior on the boundary  $\partial\Omega$  is described by Neumann conditions  $\partial_{\nu} u = 0$  with  $\nu$  denoting the outward unit normal to  $\partial\Omega$ . Given any fixed distribution u, suppose that both death rate  $\mu(u, \cdot)$  and parameter-dependent birth rate  $nb(u, \cdot)$  are functions of age only. Then time-independent (i.e. equilibrium) solutions to the corresponding evolution problem satisfy the equations

$$\partial_a u - \operatorname{div}_x \left( \mathfrak{a}(U(x)) \nabla_x u \right) + \mu(u, a) u = 0 , \quad a \in (0, a_m) , \quad x \in \Omega ,$$
(1.12)

$$u(0,x) = n \int_0^{a_m} b(u,a)u(a,x) \,\mathrm{d}a \,, \quad x \in \Omega \,, \tag{1.13}$$

$$\partial_{\nu} u(a,x) = 0$$
,  $a \in (0,a_m)$ ,  $x \in \partial \Omega$ , (1.14)

$$U(x) = \int_0^{a_m} u(a, x) \,\mathrm{d}a \,, \quad x \in \Omega \,.$$
 (1.15)

The goal is to show that this parameter-dependent problem admits a branch of solutions (n, u) with  $u \ge 0$  but  $u \ne 0$  intersecting at some point with the trivial branch  $(n, u) = (n, 0), n \in \mathbb{R}$ .

To do so we first reformulate the problem within the framework of semigroup theory. Given a suitable function u set  $U := \int_0^{a_m} u(a, \cdot) da$  and define a linear operator  $A(u) : W_{p,\mathcal{B}}^2(\Omega) \to L_p(\Omega)$  by

$$A(u)v := -\operatorname{div}_x(\mathfrak{a}(U)\nabla_x v) , \quad v \in W^2_{p,\mathcal{B}}(\Omega) := \{\phi \in W^2_p(\Omega) ; \partial_\nu \phi = 0\},$$

where  $E_0 := L_p(\Omega)$  is the space of *p*-integrable functions on  $\Omega$  and  $W_p^2(\Omega)$  is the usual Sobolev space of order 2 over  $L_p(\Omega)$ . This notation allows us to write (1.12)-(1.15) abstractly as  $E_0$ -valued equations for

 $u: J \to E_0$  of the form

$$\partial_a u + A(u)u + \mu(u, a)u = 0, \quad a \in (0, a_m),$$
  
$$u(0) = n \int_0^{a_m} b(u, a)u(a) \, da.$$
 (1.16)

Let  $\{e^{-aA(u)}; a \ge 0\}$  denote the analytic semigroup on  $E_0 = L_p(\Omega)$  associated with -A(u); that is, given any function  $v^0 \in L_p(\Omega)$  let  $v(a) := e^{-aA(u)}v^0$  be the unique classical solution to

 $\partial_a v + A(u)v = 0, \quad a \in (0, a_m) , \quad v(0) = v^0 .$ 

Then any solution to (1.12), (1.14), (1.15) must satisfy the relation

$$u(a) = e^{-\int_0^a \mu(u,r) dr} e^{-aA(u)} u(0) , \quad a \in [0, a_m) .$$

Using the formula for u(0) in (1.16) we obtain that u(0) satisfies

l

$$u(0) = n \int_0^{a_m} b(u, a) \, e^{-\int_0^a \mu(u, r) \mathrm{d}r} \, e^{-aA(u)} \, \mathrm{d}a \, u(0) \,. \tag{1.17}$$

In particular, u(0) is an eigenvector to the eigenvalue 1/n of the linear operator

$$Q_u := \int_0^{a_m} b(u, a) \, e^{-\int_0^a \mu(u, r) \mathrm{d}r} \, e^{-aA(u)} \, \mathrm{d}a \,. \tag{1.18}$$

Classical Sobolev embedding theorems and the parabolic maximum principle ensure that, for a > 0 and u fixed, the linear operator  $e^{-aA(u)}$  is compact and strongly positive considered as an operator on a suitable subspace of  $L_p(\Omega)$  (that is, on an interpolation space between  $L_p(\Omega)$  and  $W^2_{p,\mathcal{B}}(\Omega)$ ). It then follows that also the operator  $Q_u$  possesses these properties of being a compact and strongly positive operator on this subspace of  $L_p(\Omega)$ . Consequently, the Krein-Rutman theorem implies in particular that the spectral radius of  $Q_0$  (i.e. of  $Q_u$  with  $u \equiv 0$  in (1.18)) is a simple eigenvalue with an eigenfunction B belonging to the interior of this subspace. This last property is crucial for deriving positive solutions to (1.12)-(1.15). Indeed, we first investigate the linearization of (1.16) around  $u \equiv 0$ , i.e. the problem

$$\partial_a u + A(0)u + \mu(0, a)u = h_2(a) , \quad a \in (0, a_m)$$
$$u(0) - \int_0^{a_m} b(0, a)u(a) \, \mathrm{d}a = h_1 ,$$

for  $h_1$ ,  $h_2$  given, and prove that the corresponding solution operator is a Fredholm operator of index zero by using the property of maximal regularity of the operator  $A(0) + \mu(0, \cdot)$  (see Section 2 for a definition and details). This allows us to tackle the nonlinear problem (1.16) by the implicit function theorem (see [7]), which yields a nontrivial branch of solutions  $(n, u) = (n_{\varepsilon}, u_{\varepsilon}), 0 \le \varepsilon < \varepsilon_0$ , of the form

$$u_{\varepsilon}(a) = \varepsilon \left( e^{-\int_0^a \mu(0,r) dr} e^{-aA(0)} B + z_{\varepsilon} \right) \quad (z_{\varepsilon} \text{ being a suitable perturbation})$$

emanating from the trivial branch  $(n, u) = (n, 0), n \in \mathbb{R}$  at a certain critical point (for convenience chosen to be (n, u) = (1, 0)). Since  $u_{\varepsilon}$  stays in a neighborhood of B for  $\varepsilon > 0$  small, the strict positivity of B ensures the positivity of  $u_{\varepsilon}$ .

One can also deduce more information about the direction of bifurcation. For instance, suppose that, for all age classes, the smallest death rate and the largest fertility rate occur at the lowest population densities, that is, let  $\mu(u, a) \ge \mu(0, a)$  and  $b(u, a) \le b(0, a)$  for all  $a \in (0, a_m)$  and all densities u. Then necessarily  $n_{\varepsilon} \ge 1$  as can be derived easily from (1.17) by observing that movement alone does not alter the number of individuals. Thus bifurcation must be "to the right", i.e. supercritical bifurcation occurs.

As pointed out already, arguments provided in [8] for the spatially homogeneous case indicate that the trivial solution looses stability when the parameter value n increases through the critical value n = 1 and – in the case of supercritical bifurcation – the nontrivial branch (at least near the critical point (1,0)) consists of stable equilibria. Though this exchange of stability is not proved herein for the case with nonlinear

diffusion and thus remains a conjecture, applied to the present example its biological interpretation would be that low fertility rates lead to extinction and high fertility rates allow for survival of the population.

### 2. Abstract Formulation

Given Banach spaces E and F we write  $\mathcal{L}(E, F)$  for the space of bounded linear operators from E to F equipped with the usual operator norm, and we put  $\mathcal{L}(E) := \mathcal{L}(E, E)$ . We write r(A) for the spectral radius of  $A \in \mathcal{L}(E)$ . If A is a closed linear operator in E we let  $s(A) := \sup\{\operatorname{Re}\lambda; \lambda \text{ is an eigenvalue of } A\}$  denote the spectral bound of A. The subspace of  $\mathcal{L}(E)$  consisting of compact operators is  $\mathcal{K}(E)$ . If E is an ordered Banach space we write  $\mathcal{L}_+(E)$  and  $\mathcal{K}_+(E)$  for the corresponding positive operators. We let  $\mathcal{L}is(E, F)$  denote the subspace of  $\mathcal{L}(E, F)$  of topological linear isomorphisms. If  $E \stackrel{d}{\hookrightarrow} F$ , that is, if E is densely embedded in F, then  $\mathcal{H}(E, F)$  is the set of all negative generators of analytic semigroups on F with domain E.  $\mathcal{BIP}(E; \phi)$  stands for the set of operators with bounded imaginary powers and power angle  $\phi \in [0, \pi/2)$ , that is, those linear operators A in E for which there is  $M \ge 1$  such that  $||A^{it}||_{\mathcal{L}(E)} \le Me^{\phi|t|}$ ,  $t \in \mathbb{R}$ . For details we refer to [2].

Throughout this paper we suppose that  $E_0$  is a real Banach space and  $E_1 \stackrel{d}{\longleftrightarrow} E_0$ , that is,  $E_1$  is a densely and compactly embedded subspace of  $E_0$ . We fix  $p \in (1, \infty)$ , put  $\varsigma := \varsigma(p) := 1 - 1/p$  and set  $E_{\varsigma} := (E_0, E_1)_{\varsigma,p}$  with  $(\cdot, \cdot)_{\varsigma,p}$  being the real interpolation functor. Similarly we choose for each  $\alpha \in (0,1) \setminus \{1-1/p\}$  an arbitrary admissible interpolation functor  $(\cdot, \cdot)_{\alpha}$  and put  $E_{\alpha} := (E_0, E_1)_{\alpha}$  so that  $E_1 \stackrel{d}{\longleftrightarrow} E_{\alpha}$  (see [2]). If  $E_0$  is ordered by a closed convex cone  $E_0^+$ , then the interpolation spaces are equipped with the order naturally induced by  $E_0^+$ . Given  $a_m \in (0, \infty]$  we set  $J := [0, a_m)$  which thus may be bounded or unbounded. Moreover, we put

$$\mathbb{E}_0 := L_p(J, E_0) , \qquad \mathbb{E}_1 := L_p(J, E_1) \cap W_p^1(J, E_0)$$

and recall that

$$\mathbb{E}_1 \hookrightarrow BUC(J, E_{\varsigma}) \tag{2.1}$$

according to, e.g. [2, III.Thm.4.10.2], where BUC stands for bounded and uniformly continuous. In particular, the trace  $\gamma u := u(0)$  is well-defined for  $u \in \mathbb{E}_1$ .

We then study problems of the form

$$\partial_a u + \mathbb{A}(u, a) u = 0, \qquad a \in J, \tag{2.2}$$

$$u(0) = n\ell(u) \tag{2.3}$$

where  $\mathbb{A}(u, a) \in \mathcal{L}(E_1, E_0)$  and  $\ell(u) \in E_{\varsigma}$  for  $a \in J$  and  $u \in \mathbb{E}_1$  with  $\ell(0) = 0$ . We will impose more restrictions later. Introducing  $\mathbb{A}_0(a) := \mathbb{A}(0, a)$  and assuming a decomposition

$$\ell(u) = \ell_0(u) + \ell_*(u)$$

with a linear part  $\ell_0$ , we first focus our attention on the linearization around 0 of the above problem and show that the corresponding solution operator is a Fredholm operator of index zero.

## 2.1. Preliminaries. In the following we assume that

$$\ell_0 \in \mathcal{L}(\mathbb{E}_1, E_\vartheta) \quad \text{for some} \quad \vartheta \in (\varsigma, 1)$$

$$(2.4)$$

and that

$$\mathbb{A}_{0} \in L_{\infty}(J, \mathcal{L}(E_{1}, E_{0})) \text{ generates a parabolic evolution operator}$$
$$\Pi_{0}(a, \sigma), 0 \leq \sigma \leq a < a_{m}, \text{ on } E_{0} \text{ with regularity subspace } E_{1} \text{ and}$$
(2.5) possesses maximal  $L_{p}$ -regularity, that is,  $(\partial_{a} + \mathbb{A}_{0}, \gamma) \in \mathcal{L}is(\mathbb{E}_{1}, \mathbb{E}_{0} \times E_{c})$ .

For details about evolution operators and operators possessing maximal regularity we refer the reader, e.g., to [2]. It seems to be worthwhile to point out that, owing to (2.5) and [2, III.Prop.1.3.1], the problem

$$\partial_a u + \mathbb{A}_0(a) u = f(a), \quad a \in J, \qquad u(0) = u^0$$

admits for each datum  $(f, u^0) \in \mathbb{E}_0 \times E_{\varsigma}$  a unique solution  $u \in \mathbb{E}_1$  given by

$$u(a) = \Pi_0(a,0)u^0 + \int_0^a \Pi_0(a,\sigma)f(\sigma) \,\mathrm{d}\sigma \,, \quad a \in J \,,$$
(2.6)

satisfying for some  $c_0 > 0$ 

$$||u||_{\mathbb{E}_1} \le c_0 \left( ||f||_{\mathbb{E}_0} + ||u^0||_{E_\varsigma} \right) \,. \tag{2.7}$$

In particular,

$$\Pi_0(\cdot, 0) \in \mathcal{L}(E_{\varsigma}, \mathbb{E}_1) \quad \text{and} \quad K_0 \in \mathcal{L}(\mathbb{E}_0, \mathbb{E}_1) ,$$
(2.8)

where

$$(K_0 f)(a) := \int_0^a \Pi_0(a, \sigma) f(\sigma) \, \mathrm{d}\sigma \,, \qquad a \in J \,, \quad f \in \mathbb{E}_0 \,,$$
$$(f \mapsto \ell_0(K_0 f)) \in \mathcal{L}(\mathbb{E}_0, E_\varsigma) \,. \tag{2.9}$$

while, due to (2.4),

$$(f = 0, (0, 0), (0, 0), (0, 0)$$
 and the fact that  $E_{1,2}(c_{1,2}, E_{2,3}, c_{2,3}, c_{2,3}, c_{3,3}, c_{3,$ 

Moreover, we obtain from (2.4), (2.8), and the fact that 
$$E_{\vartheta} \hookrightarrow E_{\varsigma}$$
 (e.g. see [2, I.Thm.2.11.1])

$$Q_0 \in \mathcal{L}(E_{\varsigma}, E_{\vartheta}) \cap \mathcal{K}(E_{\varsigma}) \quad \text{for} \quad Q_0 w := \ell_0 \big( \Pi_0(\cdot, 0) w \big) \,, \quad w \in E_{\varsigma} \,.$$
(2.10)

The next result will be fundamental for what follows.

Lemma 2.1. Suppose (2.4) and (2.5). Then the operator

$$Lu := \left(\gamma u - \ell_0(u), (\partial_a + \mathbb{A}_0)u\right)$$

satisfies  $L \in \mathcal{L}(\mathbb{E}_1, E_{\varsigma} \times \mathbb{E}_0)$  and has a closed kernel ker(L) and a closed range rg(L) of finite dimension and codimension, respectively, both of which admit bounded projections  $P_k$  and  $P_r$ . In fact,

$$\begin{aligned} &\ker(L) = \operatorname{span} \{ \Pi_0(\cdot, 0) w \, ; \, w \in \ker(1 - Q_0) \} \, , \\ &\operatorname{rg}(L) = \{ (h_1, h_2) \in E_{\varsigma} \times \mathbb{E}_0 \, ; \, h_1 + \ell_0(K_0 h_2) \in \operatorname{rg}(1 - Q_0) \} \, , \\ & E_{\varsigma} \times \mathbb{E}_0 = \operatorname{rg}(L) \oplus (N \times \{0\}) \, , \end{aligned}$$

where  $E_{\varsigma} = \operatorname{rg}(1 - Q_0) \oplus N$ , and

$$\dim(\ker(L)) \le \operatorname{codim}(\operatorname{rg}(L)) = \dim(\ker(1-Q_0)) < \infty$$

*Proof.* First observe that (2.6) implies that, for  $(h_1, h_2) \in E_{\varsigma} \times \mathbb{E}_0$ , the equation  $Lu = (h_1, h_2)$  with  $u \in \mathbb{E}_1$  is equivalent to

$$u = \Pi_0(\cdot, 0)u(0) + K_0h_2, \quad (1 - Q_0)u(0) = h_1 + \ell_0(K_0h_2).$$
(2.11)

If 1 is not an eigenvalue of  $Q_0 \in \mathcal{K}(E_{\varsigma})$ , then (2.11) easily entails that ker(L) is trivial. Moreover, in this case we have  $h_1 + \ell_0(K_0h_2) \in E_{\varsigma}$  for any  $(h_1, h_2) \in E_{\varsigma} \times \mathbb{E}_0$  by (2.9) and there is a unique  $w \in E_{\varsigma}$  for which  $(1 - Q_0)w = h_1 + \ell_0(K_0h_2)$ . Thus  $u := \Pi_0(\cdot, 0)w + K_0h_2$  belongs to  $\mathbb{E}_1$  due to (2.8) and satisfies  $Lu = (h_1, h_2)$ , whence  $\operatorname{rg}(L) = E_{\varsigma} \times \mathbb{E}_0$  from which the claim follows in this case.

Otherwise, if 1 is an eigenvalue of  $Q_0 \in \mathcal{K}(E_{\varsigma})$ , then (2.11) ensures

$$\ker(L) = \operatorname{span}\left\{\Pi_0(\cdot, 0)w; w \in \ker(1 - Q_0)\right\} \subset \mathbb{E}_1$$

which is clearly closed since  $L \in \mathcal{L}(\mathbb{E}_1, E_{\varsigma} \times \mathbb{E}_0)$  by (2.4), (2.5). In particular, the dimension of ker(L) does not exceed the dimension of ker $(1 - Q_0)$ , the latter clearly being finite since the eigenvalue 1 has finite multiplicity. Therefore, ker(L) is complemented in  $\mathbb{E}_1$  and admits a bounded projection  $P_k \in \mathcal{L}(\mathbb{E}_1, \text{ker}(L))$ (e.g., see [24, Lem.4.21]). Next, given  $(h_1, h_2) \in \text{rg}(L) \subset E_{\varsigma} \times \mathbb{E}_0$  and  $u \in \mathbb{E}_1$  with  $Lu = (h_1, h_2)$ , we have  $h_1 + \ell_0(K_0h_2) \in \operatorname{rg}(1 - Q_0)$  as observed in (2.11). Conversely, if  $(h_1, h_2) \in \mathbb{E}_{\varsigma} \times \mathbb{E}_0$  and  $(1 - Q_0)w = h_1 + \ell_0(K_0h_2)$  for some  $w \in \mathbb{E}_{\varsigma}$ , then  $Lu = (h_1, h_2)$  for  $u := \Pi_0(\cdot, 0)w + K_0h_2$ . Thus

$$\operatorname{rg}(L) = \left\{ (h_1, h_2) \in E_{\varsigma} \times \mathbb{E}_0 ; \, h_1 + \ell_0(K_0 h_2) \in \operatorname{rg}(1 - Q_0) \right\} \,.$$
(2.12)

Since  $Q_0$  is compact,  $M := \operatorname{rg}(1 - Q_0)$  is a closed subspace of  $E_{\varsigma}$ , hence  $\operatorname{rg}(L)$  is closed in  $E_{\varsigma} \times \mathbb{E}_0$  due to (2.9). Furthermore,  $\operatorname{codim}(M) = \dim(\ker(1 - Q_0)) < \infty$ , and hence M is complemented in  $E_{\varsigma}$ , that is,  $E_{\varsigma} = M \oplus N$ . Let  $P_M \in \mathcal{L}(E_{\varsigma})$  denote the projection onto M and set

$$P_r(h_1, h_2) := \left( P_M h_1 - (1 - P_M) \ell_0(K_0 h_2), h_2 \right).$$
(2.13)

Then clearly  $P_r^2 = P_r \in \mathcal{L}(E_{\varsigma} \times \mathbb{E}_0)$  by (2.9),  $P_r(E_{\varsigma} \times \mathbb{E}_0) = \operatorname{rg}(L)$  by (2.12), and

$$(1 - P_r)(h_1, h_2) = ((1 - P_M)(h_1 + \ell_0(K_0 h_2)), 0) \in N \times \{0\}.$$

Thus we conclude that  $E_{\varsigma} \times \mathbb{E}_0 = \operatorname{rg}(L) \oplus (N \times \{0\})$  and so

$$\operatorname{codim}(\operatorname{rg}(L)) = \dim(N) = \dim(\ker(1 - Q_0))$$

This proves the assertion.

The verification of (2.5) is not a simple task in general. We thus recall conditions that allow us in Section 3 to consider cases for which (2.5) is readily verified.

## Lemma 2.2. Suppose that

$$\mathbb{A}_0 \in BUC(J, \mathcal{L}(E_1, E_0))$$
 generates a parabolic evolution operator on  $E_0$ . (2.14)

Further suppose, for each  $a \in J$ , that 0 belongs to the resolvent set of  $\mathbb{A}(a)$ , that

$$A_0(a)$$
 possesses maximal  $L_p$ -regularity, (2.15)

and that

$$\lim_{a \to \infty} \mathbb{A}_0(a) \text{ exists in } \mathcal{L}(E_1, E_0) \text{ if } a_m = \infty .$$
(2.16)

Then (2.5) is satisfied.

Proof. This is a consequence of [25, Thm.1.4].

**Remarks 2.3.** (a) If  $\mathbb{A}_0 \in C^{\rho}(J, \mathcal{H}(E_1, E_0))$  for some  $\rho > 0$ , then it generates a parabolic evolution operator on  $E_0$  due to [2, II.Cor.4.4.2].

(b) In case that  $E_0$  is a UMD-space (see [2] for a definition and properties), condition (2.15) holds if for each  $a \in J$  there is some angle  $\theta(a) \in [0, \pi/2)$  for which  $\mathbb{A}_0(a) \in \mathcal{BIP}(E_0; \theta(a))$ , see [2, III.Thm.4.10.7].

2.2. Nonlinear Theory. We now focus on problem (2.2), (2.3). Let  $m \in \mathbb{N} \setminus \{0\}$  and let  $\Sigma$  denote an open ball in  $\mathbb{E}_1$  centered at 0 of some positive radius  $R_0 > 0$ . Suppose that

$$\mathbb{A} \in C^m \big( \Sigma, L_\infty(J, \mathcal{L}(E_1, E_0)) \big) \quad \text{and} \quad \mathbb{A}_0 := \mathbb{A}(0) \text{ satisfies (2.5)} .$$
(2.17)

We set  $\mathbb{A}_*(u) := \mathbb{A}(u) - \mathbb{A}_0$  and sometimes write  $\mathbb{A}(u, a) := \mathbb{A}(u)(a)$  for  $u \in \Sigma$ ,  $a \in J$  and accordingly  $\mathbb{A}_*(u, a) := \mathbb{A}_*(u)(a)$ . We also assume that  $\ell$  admits a decomposition

$$\ell(u) = \ell_0(u) + \ell_*(u) , \qquad (2.18)$$

where the linear part  $\ell_0$  satisfies (2.4) and  $\ell_*$  is such that  $\ell_*(\varepsilon u) = \varepsilon \overline{\ell}_*(\varepsilon, u), u \in \Sigma, |\varepsilon| < 1$ , for some function

$$\bar{\ell}_* \in C^m((-1,1) \times \Sigma, E_\varsigma) \quad \text{with} \quad \bar{\ell}_*(0,\cdot) = 0 , \quad D_u \bar{\ell}_*(0,\cdot) = 0 .$$
 (2.19)

We put

$$T(\lambda, u) := \lambda \big( \ell_0(u), 0 \big) + \big( (\lambda + 1) \ell_*(u), -\mathbb{A}_*(u) u \big) , \quad (\lambda, u) \in \mathbb{R} \times \Sigma ,$$

and note that with  $n = \lambda + 1$  problem (2.2), (2.3) can be be re-written as  $Lu = T(\lambda, u)$  with L being given in Lemma 2.1. We introduce  $\overline{T} \in C^m(\mathbb{R} \times (-1, 1) \times \Sigma, E_{\varsigma} \times \mathbb{E}_0)$  as

$$\bar{T}(\lambda,\varepsilon,u) := \lambda \big( \ell_0(u), 0 \big) + \big( (\lambda+1) \bar{\ell}_*(\varepsilon,u), -\mathbb{A}_*(\varepsilon u) u \big) \,, \quad (\lambda,\varepsilon,u) \in \mathbb{R} \times (-1,1) \times \Sigma \,,$$

and observe that  $T(\lambda, \varepsilon u) = \varepsilon \overline{T}(\lambda, \varepsilon, u)$ . Nontrivial solutions to (2.2), (2.3) are then provided by the following

**Theorem 2.4.** Suppose (2.4), (2.17), (2.18), and (2.19). Moreover, suppose that  $r(Q_0) = 1$  is an eigenvalue of  $Q_0 \in \mathcal{K}(E_{\varsigma})$  with geometric multiplicity 1, where  $Q_0$  is defined in (2.10), and let  $B \in E_{\varsigma}$  be a corresponding eigenvector. Then there exists  $\varepsilon_0 > 0$  such that the problem

$$\partial_a u + \mathbb{A}(u, a) u = 0, \qquad a \in J$$
  
 $u(0) = n \ell(u)$ 

has a branch of nontrivial solutions  $\{(n_{\varepsilon}, u_{\varepsilon}) \in \mathbb{R}^+ \times \mathbb{E}_1; 0 < |\varepsilon| < \varepsilon_0\}$  of the form

$$n_{\varepsilon} = 1 + \kappa_{\varepsilon}$$
,  $u_{\varepsilon} = \varepsilon (\Pi_0(\cdot, 0)B + z_{\varepsilon})$ ,  $0 < |\varepsilon| < \varepsilon_0$ ,

where  $[\varepsilon \mapsto \kappa_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$  and  $[\varepsilon \mapsto z_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \ker(P_k)$  are *m*-times continuously differentiable with  $\kappa_0 = 0$  and  $z_0 = 0$ , where  $P_k \in \mathcal{L}(\mathbb{E}_1)$  is the projection onto  $\ker(L)$ .

*Proof.* We re-write (2.2), (2.3) as  $Lu = T(\lambda, u)$  and validate the requirements for Theorem 1 in [7]. First recall that Lemma 2.1 warrants that  $L \in \mathcal{L}(\mathbb{E}_1, E_{\varsigma} \times \mathbb{E}_0)$  has a closed kernel ker $(L) = \text{span}\{\Pi_0(\cdot, 0)B\}$  and a closed range rg(L) both admitting bounded projections  $P_k$  and  $P_r$ , respectively, and that the codimension of rg(L) equals 1. Thus H1 and H2 in [7] hold. To validate H3 therein we just have to observe that for  $y \in \text{ker}(L) \cap \Sigma$ 

$$T(0,0,y) = (\ell_*(0,y), -\mathbb{A}_*(0)y) = (0,0)$$

and

$$D_3\bar{T}(0,0,y) = (0, -\mathbb{A}_*(0)y) = (0,0)$$

It remains to verify H4 in [7]. For, let  $1 - P_r$  be the projection of  $E_{\varsigma} \times \mathbb{E}_0$  onto the one-dimensional space  $N \times \{0\}$  (see Lemma 2.1) and let  $c(\lambda, \varepsilon, z)$  be the component of  $\overline{T}(\lambda, \varepsilon, \Pi_0(\cdot, 0)B + z)$  with respect to the basis  $\{(B, 0)\}$  of  $N \times \{0\}$  for given  $\lambda \in \mathbb{R}$ ,  $|\varepsilon| < 1$ , and  $||z||_{\mathbb{E}_1} < R_0/2$ . Here we may assume without loss of generality that  $||\Pi_0(\cdot, 0)B||_{\mathbb{E}_1} < R_0/2$ . Hence it follows from  $Q_0B = B \in N$ , (2.19), and (2.13) that

$$(1 - P_r)\overline{T}(\lambda, 0, \Pi_0(\cdot, 0)B) = (1 - P_r)(\lambda B, 0) = \lambda(B, 0) ,$$

that is,  $c_{\lambda}(\lambda, 0, 0) = 1$ . Now [7, Thm.1] implies the assertion.

**Remark 2.5.** Clearly, the result applies to non-homogeneous problems

$$\partial_a u + \mathbb{A}(u, a) u = g(u, a), \qquad a \in J,$$
  
 $u(0) = n \ell(u)$ 

as well provided that there is  $\bar{g}$  such that  $g(\varepsilon u, \cdot) = \varepsilon \bar{g}(\varepsilon, u, \cdot)$  with

$$[(\varepsilon, u) \mapsto \overline{g}(\varepsilon, u, \cdot)] \in C^m((-1, 1) \times \Sigma, \mathbb{E}_0)$$
 and  $\overline{g}(0, \cdot, \cdot) = 0$ .

Next we compute the  $\varepsilon$ -expansion of the branch  $(n_{\varepsilon}, u_{\varepsilon})$ . Under the assumptions of Theorem 2.4 let  $P_k \in \mathcal{L}(\mathbb{E}_1)$  denote the projection onto  $\ker(L) = \operatorname{span}\{\Pi_0(\cdot, 0)B\}$  such that  $P_k u = \xi(u)\Pi_0(\cdot, 0)B$  with  $\xi(u) \in \mathbb{R}$  for  $u \in \mathbb{E}_1$ . Again we set  $M = \operatorname{rg}(1 - Q_0)$  and  $E_{\varsigma} = M \oplus N$  with corresponding projection  $P_M$  (see the proof of Lemma 2.1).

**Proposition 2.6.** In addition to the assumptions of Theorem 2.4 with  $m \ge 2$  suppose that  $D_u \ell_*(0) = 0$ . Then the branch of nontrivial solutions  $(n_{\varepsilon}, u_{\varepsilon}), |\varepsilon| < \varepsilon_0$ , from Theorem 2.4 can be written in the form

$$n_{\varepsilon} = 1 + \zeta \varepsilon + \bar{n}_{\varepsilon}$$
,  $u_{\varepsilon} = \varepsilon \Pi_0(\cdot, 0)B + \varepsilon^2 (\Pi_0(\cdot, 0)\xi - K_0h) + \varepsilon \bar{u}_{\varepsilon}$ 

for  $|\varepsilon| < \varepsilon_0$ , where  $[\varepsilon \mapsto \bar{n}_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$  and  $[\varepsilon \mapsto \bar{u}_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \ker(P_k)$  are such that  $|\bar{n}_{\varepsilon}| = o(\varepsilon^2)$  and  $||\bar{u}_{\varepsilon}||_{\mathbb{E}_1} = o(\varepsilon^2)$  as  $|\varepsilon| \to 0$ . The function  $h \in \mathbb{E}_0$  is defined by

$$h(a) := (D_u \mathbb{A}_*(0)(\Pi_0(\cdot, 0)B)(a))\Pi_0(a, 0)B, \quad a \in J,$$

 $\zeta \in \mathbb{R}$  is the unique coefficient of

$$(1 - P_M)(\ell_0(K_0h) - g) = \zeta B \in N$$

with

$$g := \frac{1}{2} D_u^2 \ell_*(0) [\Pi_0(\cdot, 0) B, \Pi_0(\cdot, 0) B] \in E_{\varsigma} ,$$

and  $\xi \in E_{\varsigma}$  is the unique solution to

$$(1-Q_0)\xi = \zeta B + g - \ell_0(K_0h) \in M$$
,  $\xi(\Pi_0(\cdot,0)\xi) = \xi(K_0h)$ .

*Proof.* We plug the twice continuously differentiable functions  $\kappa = \kappa_{\varepsilon}$  and  $u = u_{\varepsilon}$  provided by Theorem 2.4 into the equation  $Lu = T(\lambda, u)$ , which we then differentiate twice with respect to  $\varepsilon$ . Evaluating the result at  $\varepsilon = 0$  and using  $D_{\mu}\ell_*(0) = 0$  together with  $\ell_0(\Pi_0(\cdot, 0)B) = B$ , we obtain

$$Lz'_{0} = (\kappa'_{0}B + g, -h)$$
(2.20)

with dashes denoting derivatives with respect to  $\varepsilon$  and g, h as given in the statement. Hence, from (2.11),

$$y := \kappa'_0 B + g - \ell_0(K_0 h) \in M$$

and thus, since  $P_M B = 0$ ,

$$(1 - P_M)(-g + \ell_0(K_0h)) = \kappa'_0 B$$

from which the formula for  $n_{\varepsilon}$  follows by setting  $\zeta := \kappa'_0$ . Next, if  $\varrho \in E_{\zeta}$  is an arbitrarily fixed solution to  $(1 - Q_0)\varrho = y$ , then any other  $\eta \in E_{\varsigma}$  with  $(1 - Q_0)\eta = y$  can be written uniquely in the form  $\eta_{\alpha} := \eta = \varrho + \alpha B$  for some  $\alpha \in \mathbb{R}$ . Writing  $w := z'_0 \in \mathbb{E}_1$  we have  $w = \Pi_0(\cdot, 0)\eta_{\alpha} - K_0 h$  by (2.20) and (2.11) with  $\alpha \in \mathbb{R}$  determined by the constraint that w must belong to ker $(P_k)$ . This is obtained by observing that

$$0 = P_k w = \left( \xi(\Pi_0(\cdot, 0)\varrho) + \alpha - \xi(K_0 h) \right) \Pi_0(\cdot, 0) B ,$$

that is,  $\alpha = k(K_0h) - k(\Pi_0(\cdot, 0)\varrho)$ . For this  $\alpha$  we put  $\xi := \eta_\alpha$  and get  $u_{\varepsilon} = \varepsilon \Pi_0(\cdot, 0) B + \varepsilon^2 (\Pi_0(\cdot, 0) \xi - K_{\varepsilon} h)$ 

$$u_{\varepsilon} = \varepsilon \Pi_0(\cdot, 0)B + \varepsilon^2 (\Pi_0(\cdot, 0)\xi - K_0h) + \varepsilon \bar{u}_{\varepsilon}$$

with  $\|\bar{u}_{\varepsilon}\|_{\mathbb{E}_1} = o(\varepsilon^2)$  as  $|\varepsilon| \to 0$ .

2.3. **Positive Solutions.** We shall give conditions under which the nontrivial equilibrium solutions are positive. To this end we suppose that

$$E_0$$
 is ordered by a closed convex cone  $E_0^+$ . (2.21)

Then the interpolation spaces  $E_{\sigma}$  are given their natural order induced by the cone  $E_{\sigma}^+ := E_{\sigma} \cap E_0^+$ . For information on positive and strongly positive operators we refer to [11, 26]. If (n, u) is a solution to (2.2), (2.3) we say that u is a positive equilibrium provided that  $u(a) \in E_0^+$  for  $a \in J$ .

Before turning to positive solutions we remark the following about the assumptions on  $Q_0$  in Theorem 2.4.

**Remark 2.7.** Assume that the parabolic evolution operator  $\Pi_0$  corresponding to  $\mathbb{A}_0$  in (2.5) is positive, that is,  $\Pi_0(a,\sigma) \in \mathcal{L}_+(E_0)$  for  $0 \leq \sigma \leq a < a_m$ . If also  $\ell_0 \in \mathcal{L}_+(\mathbb{E}_1, E_\vartheta)$  in (2.4), then  $Q_0 \in \mathcal{K}_+(E_\varsigma)$ and thus the Krein-Rutman theorem entails that the spectral radius  $r(Q_0)$  is (if nonzero) an eigenvalue of finite multiplicity with a positive eigenvector  $B \in E_{\varsigma}^+$ . Hence, in this case the assumption in Theorem 2.4 that the normalized spectral radius  $r(Q_0) = 1$  is an eigenvalue is no severe restriction. More restrictive is the assumption that this eigenvalue has geometric multiplicity 1. However, if  $Q_0$  is strongly positive or irreducible, then  $r(Q_0) = 1$  is simple, see for instance [11, Sect.12] or [26, App.3.2]. We also refer to the next section for concrete examples.

 $\square$ 

Proposition 2.8. Suppose the assumptions of Theorem 2.4 and (2.21). In addition,

for each 
$$u \in \Sigma$$
 let  $\mathbb{A}(u)$  generate a positive parabolic  
evolution operator  $\Pi_u(a, \sigma), \ 0 \le \sigma \le a \le a_m, \ on E_0$ . (2.22)

If  $(n_{\varepsilon}, u_{\varepsilon})$  is the branch of solutions from Theorem 2.4, then  $u_{\varepsilon}$  is positive provided that  $\varepsilon \in (0, \varepsilon_0)$  is such that

$$\frac{1}{\varepsilon}\gamma u_{\varepsilon} = B + \gamma z_{\varepsilon} \in E_{\varsigma}^{+} .$$
(2.23)

In particular, if B belongs to the interior of  $E_{\varsigma}^+$ , then  $u_{\varepsilon}$  is positive for  $\varepsilon > 0$  sufficiently small.

*Proof.* This follows from the fact that under the stated assumptions any solution (n, u) to (2.2), (2.3) satisfies

$$u(a) = \Pi_u(a,0)u(0) , \quad a \in J$$

hence  $u(a) \in E_{\varsigma}^+$  for  $a \in J$  if  $\gamma u = u(0) \in E_{\varsigma}^+$ , and from the fact that  $z_{\varepsilon} \to 0$  in  $\mathbb{E}_1 \hookrightarrow BUC(J, E_{\varsigma})$  as  $\varepsilon \to 0$ .

**Remark 2.9.** Recall that according to [2, II.Cor.4.4.2, II.Thm.6.4.2],  $\mathbb{A}(u)$  generates a positive parabolic evolution operator  $\Pi_u(a, \sigma)$  on  $E_0$  provided that  $\mathbb{A}(u) \in C^{\rho}(J, \mathcal{H}(E_1, E_0))$  for some  $\rho > 0$  and  $-\mathbb{A}(u)(a)$  is resolvent positive for each  $a \in J$ . In this case, a solution  $u \in \mathbb{E}_1$  to (2.2), (2.3) possesses additional regularity, see [2, II.Thm.1.2.1, II.Thm.5.3.1].

Under some symmetry conditions on A and  $\ell$  the equilibrium solutions provided by Theorem 2.4 are positive for each parameter value  $n_{\varepsilon}$ ,  $-\varepsilon_0 < \varepsilon < \varepsilon_0$ . More precisely, we have:

**Proposition 2.10.** Suppose the assumptions of Theorem 2.4, (2.21), and (2.22). Let  $\mathbb{A}(u) = \mathbb{A}(-u)$ and  $\ell(u) = -\ell(-u)$  for  $u \in \Sigma$ . Given  $u \in \Sigma$  set  $Q_u w := \ell(\Pi_u(\cdot, 0)w)$ ,  $w \in E_\alpha$ , and suppose that  $Q_u \in \mathcal{L}_+(E_\alpha)$  for some  $\alpha \in [0, \varsigma]$ . Moreover, suppose that any positive eigenvalue of  $Q_u$  has geometric multiplicity 1 and possesses a positive eigenvector. Then

$$C^{+} := \left\{ \left( n_{\varepsilon}, u_{\varepsilon} \right); \, \gamma u_{\varepsilon} \in E_{0}^{+} \right\} \cup \left\{ \left( n_{\varepsilon}, -u_{\varepsilon} \right); \, \gamma u_{\varepsilon} \notin E_{0}^{+} \right\}$$

consists of positive equilibria only.

*Proof.* Let  $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$ . Since  $(n_{\varepsilon}, u_{\varepsilon})$  satisfies

$$u_{\varepsilon} = \Pi_{u_{\varepsilon}}(\cdot, 0) \gamma u_{\varepsilon} , \quad \gamma u_{\varepsilon} = n_{\varepsilon} Q_{u_{\varepsilon}} \gamma u_{\varepsilon} ,$$

it follows that  $n_{\varepsilon}^{-1} > 0$  is an eigenvalue of  $Q_{u_{\varepsilon}}$  with eigenvector  $\gamma u_{\varepsilon} \in E_{\varsigma}$ . By assumption there is a corresponding positive eigenvector  $B_{u_{\varepsilon}}$  and  $\alpha_{\varepsilon} \in \mathbb{R} \setminus \{0\}$  such that  $\gamma u_{\varepsilon} = \alpha_{\varepsilon} B_{u_{\varepsilon}}$ . If  $\alpha_{\varepsilon} > 0$  then  $\gamma u_{\varepsilon} \in E_0^+$  and thus  $u_{\varepsilon}(a) \in E_0^+$  for each  $a \in J$ . Otherwise, if  $\alpha_{\varepsilon} < 0$ , then  $-u_{\varepsilon}$  is a positive equilibrium solution with parameter value  $n_{\varepsilon}$  due to  $\gamma(-u_{\varepsilon}) = -\alpha_{\varepsilon} B_{u_{\varepsilon}} \in E_0^+$  and owing to the symmetry conditions put on  $\mathbb{A}$  and  $\ell$ .

Proposition 2.8 guarantees that a branch of positive equilibria bifurcates from the branch of trivial equilibria  $(n, u) = (n, 0), n \in \mathbb{R}$ , at the critical value n = 1. Near the critical value n = 1 the set of n values corresponding to positive equilibria on the branch from Theorem 2.4 consists of n values greater (i.e. supercritical bifurcation) or less (i.e. subcritical bifurcation) than 1 depending on the sign of  $\kappa_{\varepsilon} = n_{\varepsilon} - 1$  for  $\varepsilon > 0$  sufficiently small. If  $m \ge 2$  in Theorem 2.4, this "direction of bifurcation", that is, the cases  $n_{\varepsilon} > 1$  and  $n_{\varepsilon} < 1$  for  $\varepsilon > 0$  small, depends on the sign of  $\kappa'_0 = \zeta$  (if nonzero), which in turn depends on  $(D_u(A_*(0))\Pi_0(\cdot, 0)B)$  and  $D_u^2\ell_*(0)[\Pi_0(\cdot, 0)B, \Pi_0(\cdot, 0)B]$  according to Proposition 2.6. Further, Proposition 2.10 warrants under the symmetry conditions imposed that for any of the values  $n_{\varepsilon} \neq 1$  there is a positive nontrivial equilibrium. Examples to which Propositions 2.8 and 2.10 apply will be given in the next section.

Under additional assumptions we can get more information about the positive equilibria and the direction of bifurcation. For simplicity we demonstrate this when  $\ell$  is given by

$$\ell(u) := \int_0^{a_m} b(u, a) u(a) \, \mathrm{d}a \,, \quad u \in \Sigma \,,$$
(2.24)

where  $b \in C^m(\Sigma, L_{p'}^+(J))$  with 1/p + 1/p' = 1 and b(u, a) := b(u)(a). Then (2.4), (2.18), and (2.19) clearly hold by putting

$$\ell_0(u) := \int_0^{a_m} b_0(a)u(a) \,\mathrm{d}a \,, \quad \ell_*(u) := \int_0^{a_m} b_*(u,a)u(a) \,\mathrm{d}a \tag{2.25}$$

for  $b_0(a) := b(0, a)$  and  $b_*(u, a) := b(u, a) - b_0(a)$ . Let the assumptions of Proposition 2.8 be satisfied and suppose that there exists  $\varepsilon_* \in (0, \varepsilon_0)$  such that (2.23) holds for  $\varepsilon \in (0, \varepsilon_*)$ . Let (2.22) hold and, given  $u \in \Sigma$ , assume that

$$Q_u := \int_0^{a_m} b(u, a) \Pi_u(a, 0) \,\mathrm{d}a$$

belongs to  $\mathcal{K}_+(E_{\varsigma})$ . Note that  $Q_u$  for u = 0 coincides with  $Q_0$  defined in (2.10). Set

$$N_i := \inf_{u \in \Gamma} r(Q_u) , \quad N_s := \sup_{u \in \Gamma} r(Q_u) ,$$

where  $\Gamma := \{u_{\varepsilon}; \varepsilon \in [0, \varepsilon_*)\}$ . Then  $0 \le N_i \le 1 \le N_s \le \infty$  since  $r(Q_0) = 1$ . Moreover,

$$n r(Q_u) \ge 1$$
 for  $(n, u) \in \Lambda := \{(n_\varepsilon, u_\varepsilon); \varepsilon \in [0, \varepsilon_*)\}.$  (2.26)

Indeed, given  $(n, u) \in \Lambda \setminus \{(1, 0)\}$  we have  $u(a) = \prod_u (a, 0)u(0)$  for  $a \in J$  and

$$0 \neq u(0) = n\ell(u) = nQ_u u(0) ,$$

that is, 1/n is an eigenvalue of  $Q_u \in \mathcal{L}(E_{\varsigma})$ , whence  $r(Q_u) \ge 1/n$ . Suppose in addition that

for each  $u \in \Sigma$ ,  $r(Q_u) > 0$  is the only eigenvalue of  $Q_u \in \mathcal{K}_+(E_{\varsigma})$  with positive eigenvector. (2.27)

This holds, e.g., if  $Q_u$  is strongly positive or irreducible. Then

$$n r(Q_u) = 1$$
,  $(n, u) \in \Lambda$ . (2.28)

Furthermore, letting

$$[\sigma_i, \sigma_s] := cl_{\mathbb{R}} \{ n_{\varepsilon} ; \varepsilon \in [0, \varepsilon_*) \} ,$$

it readily follows from (2.28) that

$$0 \le \sigma_i = \frac{1}{N_s} \le 1 \le \sigma_s = \frac{1}{N_i} \le \infty .$$
(2.29)

Therefore, under the assumptions of Proposition 2.8, (2.24), (2.27), and if  $r(Q_u) \leq 1$  for  $u \in \Sigma$ , we have  $N_s \leq 1$ , hence  $1 = N_s = \sigma_i$  and bifurcation must be supercritical in this case. Again, we refer to the next section for concrete examples.

### 3. APPLICATIONS TO POPULATION DYNAMICS

We now apply the obtained results to problem (2.2), (2.3). Before considering concrete diffusion operators in Examples 3.2-3.4, we first state in Example 3.1 some simple abstract conditions for A,  $\mu$ , and b under which the results of Section 2 apply. For simplicity we assume that A = A(u) does not depend explicitly on age a, i.e. we do not consider here operators of the form A = A(u, a), though the examples include an implicit local dependence A = A(u(a)) on age (see also Remark 3.5).

# 3.1. **Example.** Suppose (2.21) and that the interior $int(E_{\varsigma}^+)$ of $E_{\varsigma}^+$ is nonempty. Let

$$\ell(u) := \int_0^{a_m} b(u,a)u(a) \,\mathrm{d}a \;.$$

As observed just previous to (2.25),  $\ell$  satisfies (2.4), (2.18), and (2.19) provided that

$$b \in C^m(\Sigma, L_{p'}^+(J)), \quad b_0 := b(0) \neq 0,$$
(3.1)

for some  $m \ge 1$  and some ball  $\Sigma$  in  $\mathbb{E}_1$  centered at 0 with radius  $R_0 > 0$ . Moreover, regarding Proposition 2.6 we note that  $D_u \ell_*(0) = 0$ . Let  $\alpha \in [0, \varsigma)$  and let  $\Phi$  be the ball in  $E_\alpha$  with center 0 and radius R > 0. Let

$$A \in C^m(\Phi, \mathcal{L}(E_1, E_0)) \tag{3.2}$$

be such that

-A(w) generates an analytic semigroup on  $E_0$  and is resolvent positive for each  $w \in \Phi$ . (3.3)

Making  $R_0 > 0$  smaller if necessary it follows from the compact embedding  $E_{\varsigma} \hookrightarrow E_{\alpha}$  (see for instance [2, I.Thm.2.11.1]) and (2.1) analogously to [3, VII.Thm.6.2, VII.Thm.6.4] that the Nemitskii operator of A (again labeled A), given by

$$A(u)(a) := A(u(a)) , \quad a \in J , \quad u \in \Sigma ,$$

belongs to  $C^m(\Sigma, L_{\infty}(J, \mathcal{L}(E_1, E_0)))$ . Since  $\mathbb{E}_1 \hookrightarrow BUC^{\varsigma-\delta}(J, E_{\delta})$  for  $\delta \in [0, \varsigma)$  owing to (2.1) and the interpolation inequality in [2, I.Thm.2.11.1], we deduce from (3.3) and Remark 2.9 that

$$[a \mapsto A(u(a))] \in C^{\varsigma - \alpha}(J, \mathcal{H}(E_1, E_0))$$

generates a positive parabolic evolution operator  $U_{A(u)}(a, \sigma)$  on  $E_0$  for each  $u \in \Sigma$  in the sense of [2, II.Sect.2.1]. Set  $A_0 := A(0)$  and note that  $-A_0$  is independent of age and thus simply generates an analytic semigroup  $\{e^{-aA_0}; a \ge 0\}$  on  $E_0$ . Suppose there exist  $\omega_0 \ge 0$  and  $\phi \in [0, \pi/2)$  such that  $\omega_0 > s(-A_0)$  and

$$\omega_0 + A_0 \in \mathcal{BIP}(E_0; \phi) , \qquad (3.4)$$

where  $s(-A_0)$  denotes the spectral bound of  $-A_0$  introduced at the beginning of Section 2. Moreover, suppose that

$$e^{-aA_0} \in \mathcal{L}(E_{\varsigma})$$
 is strongly positive for  $a > 0$ , (3.5)

that is,  $e^{-aA_0}$  maps  $E_{\varsigma}^+ \setminus \{0\}$  into the interior of  $E_{\varsigma}^+$ . If  $\mu$  is a function such that

$$[u \mapsto \mu(u, \cdot)] \in C^m(\Sigma, L^+_{\infty}(J)), \qquad (3.6)$$

we set  $\mu_0(a) := \mu(0, a)$  for  $a \in J$  and further suppose that

$$\mu_0 \in BUC(J), \quad \inf_{a \in J} \mu_0(a) > \omega_0,$$
(3.7)

and

$$\lim \ \mu_0(a) \text{ exists if } a_m = \infty . \tag{3.8}$$

Put  $\mathbb{A}(u,a):=\mu(u,a)+A(u(a))$  for  $a\in J,\,u\in\Sigma$  and note that

$$\mathbb{A}_0(a) := \mathbb{A}(0, a) = \mu_0(a) + A_0, \quad a \in J$$

Clearly,  $\mathbb{A}(u, \cdot)$  generates a positive parabolic evolution operator  $\Pi_u(a, \sigma)$  on  $E_0$  for each  $u \in \Sigma$  given by

$$\Pi_u(a,\sigma) := e^{-\int_\sigma^a \mu(u,r) \mathrm{d}r} U_{A(u)}(a,\sigma) , \quad 0 \le \sigma \le a < a_m ,$$

where for  $u \equiv 0$  we have

$$\Pi_0(a,\sigma) = e^{-\int_\sigma^a \mu_0(r) \mathrm{d}r} e^{-(a-\sigma)A_0}, \quad 0 \le \sigma \le a < a_m$$

From (3.4), (3.7), and [2, III.Cor.4.8.6] it follows that we may apply Remark 2.3(b) to conclude that (2.15) holds true provided  $E_0$  is a UMD space. Lemma 2.2 then guarantees that A satisfies (2.5).

Finally, let  $Q_0 \in \mathcal{K}_+(E_\varsigma)$  be given by

$$Q_0 := \int_0^{a_m} b_0(a) \, e^{-\int_0^a \mu_0(r) \mathrm{d}r} \, e^{-aA_0} \, \mathrm{d}a$$

and note that  $Q_0 \in \mathcal{K}(E_{\varsigma})$  is strongly positive, hence irreducible (see [11, Sect.12]). Indeed, let f' be any nontrivial element of the dual cone

$$(E_{\varsigma}^{+})' := \{ f' \in E_{\varsigma}'; \langle f', h \rangle \ge 0 \text{ for all } h \in E_{\varsigma}^{+} \}$$

where  $\langle \cdot, \cdot \rangle$  denotes the duality pairing in  $E'_{\varsigma} \times E_{\varsigma}$ , and let  $h \in E^+_{\varsigma} \setminus \{0\}$ . Then  $\langle f', e^{-aA_0}h \rangle > 0$  for a > 0 since  $e^{-aA_0}h$  is an interior point of  $E^+_{\varsigma}$  due to (3.5), and thus

$$\langle f', Q_0 h \rangle = \int_0^{a_m} b_0(a) e^{-\int_0^a \mu_0(r) \mathrm{d}r} \langle f', e^{-aA_0} h \rangle \mathrm{d}a > 0 ,$$

owing to  $b_0 \neq 0$ . Hence  $Q_0 h$  is a quasi-interior point of  $E_{\varsigma}^+$  for each  $h \in E_{\varsigma}^+ \setminus \{0\}$  by definition and thus an interior point of  $E_{\varsigma}^+$  according to [6, Prop.A.2.10] since the interior of  $E_{\varsigma}^+$  is nonempty. Thus  $Q_0$  is strongly positive. In particular, it follows from the Krein-Rutman theorem [11, Thm.12.3] that  $r(Q_0) > 0$ is a simple eigenvalue of  $Q_0$  with a corresponding eigenvector  $B \in int(E_{\varsigma}^+)$  (and this is the only eigenvalue with a positive eigenvector). For convenience we assume  $b_0$  to be normalized such that  $r(Q_0) = 1$ .

Combining Lemma 2.2, Remark 2.3, Proposition 2.8, and Theorem 2.4 we obtain:

**Theorem 3.1.** Let  $E_0$  be a UMD space satisfying (2.21) and let  $int(E_{\varsigma}^+) \neq \emptyset$ . Suppose (3.1)-(3.8) and let  $b_0$  be normalized such that  $r(Q_0) = 1$ . Then the problem

$$\partial_a u + A(u(a))u + \mu(u, a)u = 0, \quad a \in J,$$
  
$$u(0) = n \int_0^{a_m} b(u, a)u(a) \,\mathrm{d}a,$$

has a branch of nontrivial solutions  $(n_{\varepsilon}, u_{\varepsilon}) \in \mathbb{R}^+ \times \mathbb{E}_1, 0 < |\varepsilon| < \varepsilon_0$ , of the form

$$u_{\varepsilon} = 1 + \kappa_{\varepsilon} , \quad u_{\varepsilon} = \varepsilon \big( \Pi_0(\cdot, 0) B + z_{\varepsilon} \big)$$

such that  $[\varepsilon \mapsto \kappa_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$  and  $[\varepsilon \mapsto z_{\varepsilon}] : (-\varepsilon_0, \varepsilon_0) \to \mathbb{E}_1$  are *m*-times continuously differentiable with  $\kappa_0 = 0$ ,  $z_0 = 0$ . If  $\varepsilon > 0$  is sufficiently small, then  $u_{\varepsilon}(a) \in E_{\varsigma}^+$  for  $a \in J$ .

If, in addition, the symmetry conditions

$$A(-u) = A(u) , \quad \mu(-u, \cdot) = \mu(u, \cdot) , \quad b(-u, \cdot) = b(u, \cdot)$$
(3.9)

hold for  $u \in \Sigma$  and if

$$Q_u := \int_0^{u_m} b(u, a) e^{-\int_0^a \mu(u, r) dr} U_{A(u)}(a, 0) da$$

for  $u \in \Sigma$  is such that  $Q_u \in \mathcal{L}_+(E_\varsigma)$  and

any positive eigenvalue of 
$$Q_u$$
 has geometric multiplicity 1  
and possesses a corresponding positive eigenvector, (3.10)

then it follows from Proposition 2.10:

**Corollary 3.2.** Suppose the assumptions of Theorem 3.1 together with (3.9), (3.10). Then, for each parameter value  $n_{\varepsilon}$  with  $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \setminus \{0\}$  provided by Theorem 3.1, there exists a positive nontrivial equilibrium solution of the form  $u_{\varepsilon}$  or  $-u_{\varepsilon}$ .

3.2. **Example.** We now consider concrete diffusion operators to which Example 3.1 apply. Let  $\Omega \subset \mathbb{R}^N$ ,  $N \geq 1$ , be a bounded and smooth domain lying locally on one side of  $\partial\Omega$ . Let  $\partial\Omega = \Gamma_0 \cup \Gamma_1$ , where  $\Gamma_0, \Gamma_1$  are both open and closed in  $\partial\Omega$  and  $\Gamma_0 \cap \Gamma_1 = \emptyset$ . Consider

$$\mathcal{A}(u,x)w := -\nabla_x \cdot \left( \mathbf{a}(u,x)\nabla_x w \right) + \mathbf{a}_1(u,x) \cdot \nabla_x w + \mathbf{a}_0(u,x)w$$

where

$$[u \mapsto a(u, \cdot)] \in C^m \left( \Phi, C^{1+\sigma}(\bar{\Omega}) \right), [u \mapsto a_1(u, \cdot)] \in C^m \left( \Phi, C^{\sigma}(\bar{\Omega}, \mathbb{R}^N) \right), \quad [u \mapsto a_0(u, \cdot)] \in C^m \left( \Phi, C^{\sigma}(\bar{\Omega}) \right),$$
(3.11)

for some  $m \ge 1$ ,  $\sigma \in (0,1)$  small, and some open ball  $\Phi$  in  $C^{1+\sigma}(\overline{\Omega})$  around 0. Moreover, assume the ellipticity condition

$$a(u,x) > 0, \quad x \in \overline{\Omega}, \quad u \in \Phi.$$
(3.12)

Let

$$\nu_0 \in C^1(\Gamma_1) \tag{3.13}$$

and let  $\nu$  denote the outward unit normal to  $\Gamma_1$ . Let

$$\mathcal{B}(x)w := \left\{ \begin{array}{ll} w \,, & \text{on } \Gamma_0 \;, \\ \frac{\partial}{\partial \nu}w + \nu_0(x)w \;, & \text{on } \Gamma_1 \;. \end{array} \right.$$

Fix  $p \in (N + 2, \infty)$  and let  $E_0 := L_p(\Omega)$  be ordered by its positive cone of functions that are nonnegative almost everywhere. Note that  $E_0$  is a UMD-space according to [2, III.Thm.4.5.2]. Set  $E_1 := W_{p,\mathcal{B}}^2(\Omega)$ , where

$$W_{p,\mathcal{B}}^{2\xi}(\Omega) := \begin{cases} W_p^{2\xi}(\Omega) , & 0 < 2\xi < 1/p , \\ \left\{ w \in W_p^{2\xi}(\Omega) ; w|_{\Gamma_0} = 0 \right\} , & 1/p < 2\xi < 1 + 1/p \\ \left\{ w \in W_p^{2\xi}(\Omega) ; \mathcal{B}w = 0 \right\} , & 2\xi > 1 + 1/p , \end{cases}$$

are subspaces of the usual Sobolev-Slobodeckii spaces  $W_p^{2\xi}(\Omega)$ . Recall that if  $E_{\xi} := (E_0, E_1)_{\xi,p}$  denotes the real interpolation space, then

$$E_{\varsigma} \doteq W_{p,\mathcal{B}}^{2\varsigma} \hookrightarrow E_{\alpha} \doteq W_{p,\mathcal{B}}^{2\alpha} \hookrightarrow C^{1+\sigma}(\bar{\Omega}) , \quad 1 + N/p + \sigma < 2\alpha < 2\varsigma := 2(1-1/p) ,$$

where dots indicate equivalent norms (see [28, 4.3.3.Thm.] for the interpolation result and [2, I.Thm.2.11.1] for the compact embedding). Also note that  $\operatorname{int}(E_{\varsigma}^+) \neq \emptyset$  (see [11, Sect.13]). We point out that  $(\mathcal{A}(u, \cdot), \mathcal{B})$  for  $u \in C^{1+\sigma}(\overline{\Omega})$  fixed is a regular elliptic boundary value problem as studied in [1]. Set

$$A(u)w := \mathcal{A}(u, \cdot)w$$
,  $w \in E_1$ ,  $u \in \Phi$ .

Then (3.11), (3.12), and [1, Sect.7,Thm.11.1] imply that -A(u) generates an analytic semigroup on  $L_p(\Omega)$  and is resolvent positive, whence (3.2) and (3.3) hold. Moreover, suppose that

$$a_0(0,x) \ge 0, \quad a_1(0,x) = 0, \quad x \in \Omega, \nu_0(x) > 0, \quad x \in \Gamma_1,$$
(3.14)

and put  $A_0 := A(0)$ . According to [1, Thm.7.1,Thm.11.1,Thm.12.1],  $-A_0$  is resolvent positive and generates a contraction semigroup on each  $L_q(\Omega)$ ,  $1 < q < \infty$ , is self-adjoint in  $L_2(\Omega)$ , and there exists a largest eigenvalue  $\lambda_0 \leq 0$  of  $-A_0 \in \mathcal{L}(E_1, E_0)$  with a positive eigenfunction  $B \in E_{\varsigma}^+$ . Moreover, [11, Cor.13.6] ensures (3.5). Hence we now are in the situation of [2, III.Ex.4.7.3(d)] and thus deduce (3.4) for each  $\omega_0 > 0$ . Given  $a_m \in (0, \infty]$  and some open ball  $\Sigma$  in  $\mathbb{E}_1 = L_p(J, W_{p, \mathcal{B}}^2(\Omega)) \cap W_p^1(J, L_p(\Omega))$  centered at 0 suppose that

$$\mu$$
 satisfies (3.6), (3.7) with  $\omega_0 = 0$ , and (3.8). (3.15)

Thus, if  $b_0 := b(0, \cdot)$  for  $b := [u \mapsto b(u, \cdot)] \in C^m(\Sigma, L^+_{p'}(J))$  is nontrivial and normalized such that

$$\int_0^{a_m} b_0(a) \, e^{-\int_0^a \mu_0(r) \mathrm{d}r} \, e^{\lambda_0 a} \, \mathrm{d}a = 1 \,, \tag{3.16}$$

then  $e^{-aA_0}B = e^{a\lambda_0}B$  for  $a \ge 0$  (being due a simple uniqueness argument) entails  $Q_0B = B$ , where

$$Q_0 = \int_0^{a_m} b_0(a) e^{-\int_0^a \mu_0(r) \mathrm{d}r} e^{-aA_0} \, \mathrm{d}a \in \mathcal{K}_+(E_\varsigma)$$

is strongly positive due to (3.5) as shown in Example 3.1. Thus  $r(Q_0) = 1$  by the Krein-Rutman theorem since  $r(Q_0)$  is the only eigenvalue with positive eigenfunction. Therefore, (3.1)-(3.8) hold and Theorem 3.1 entails:

**Proposition 3.3.** Let  $p \in (N + 2, \infty)$  and suppose (3.1), (3.11)-(3.16). Then the problem

$$\partial_a u + \mathcal{A}(u(a), x)u + \mu(u, a)u = 0, \quad a \in J, \quad x \in \Omega,$$
$$u(0, x) = n \int_0^\infty b(u, a)u(a, x) \,\mathrm{d}a, \quad x \in \Omega,$$
$$\mathcal{B}(x)u(a, x) = 0, \quad a > 0, \quad x \in \partial\Omega,$$

has a branch of nontrivial solutions

$$(n_{\varepsilon}, u_{\varepsilon}) \in \mathbb{R}^+ \times (L_p(J, W^2_{p, \mathcal{B}}(\Omega)) \cap W^1_p(J, L_p(\Omega))), \quad 0 < |\varepsilon| < \varepsilon_0,$$

of the form

$$u_{\varepsilon}(a,\cdot) = \varepsilon \left( e^{-\int_{0}^{a} \mu_{0}(r) \mathrm{d}r} e^{-aA_{0}}B + z_{\varepsilon} \right), \quad z_{\varepsilon} \in L_{p}(J, W^{2}_{p,\mathcal{B}}(\Omega)) \cap W^{1}_{p}(J, L_{p}(\Omega))$$

bifurcating from (n, u) = (1, 0), such that  $u_{\varepsilon}(a) \in L_p^+(\Omega)$  for  $a \in J$  and  $\varepsilon > 0$  small.

**Remark 3.4.** The proposition above also holds if  $E_0 := L_q(\Omega)$  and  $E_1 := W_{q,\mathcal{B}}^2(\Omega)$  for q > N + 2different from an arbitrary  $p \in (1, \infty)$ . The only difference is that the interpolation space  $E_{\varsigma}$  equals a subspace of the Besov space  $B_{q,p}^{2\varsigma}(\Omega)$  (see [28, 4.3.3.Thm.]).

3.3. **Example.** We may also consider a functional dependence of  $\mathcal{A}$  on u. Indeed, let again  $a_m \in (0, \infty]$ and let  $\Omega$ ,  $E_1$ , and  $E_0$  be as in Example 3.2 with  $p \in (N+2,\infty)$  arbitrary. Given  $u \in \mathbb{E}_1 \hookrightarrow L_1(J, L_p(\Omega))$ let  $U := \int_0^{a_m} u(a) da$  and consider  $A(u)w := \mathcal{A}(U, \cdot)w$  for  $w \in E_1 = W_{p,\mathcal{B}}^2(\Omega)$  with  $\mathcal{A}$ ,  $\mathcal{B}$  again as in Example 3.2 satisfying (3.11)-(3.14) but  $\Phi$  in (3.11) is now an open ball in  $L_p(\Omega)$  centered at 0. Suppose (3.15) with  $\inf_{a \in J} \mu(u, a) > s(-A(u))$  for  $u \in \Sigma$  and suppose  $b \in C^m(\Sigma, L_{p'}^+(J))$  with (3.16). Moreover, assume that  $b(u) \neq 0$ . Note that  $\mathbb{A}_0 := A(0) + \mu(0, \cdot)$  is exactly the same operator as in Example 3.2 and hence has the same properties as derived there. As in Example 3.1 we deduce that

$$Q_u := \int_0^{a_m} b(u, a) \, e^{-\int_0^a \mu(u, r) \mathrm{d}r} \, e^{-aA(u)} \, \mathrm{d}a \in \mathcal{K}_+(E_\varsigma)$$

is strongly positive for each  $u \in \Sigma$ , hence (2.27) holds by [11, Thm.12.3, Cor.13.6]. Exactly as in Examples 3.1 and 3.2 we obtain that all the assumptions of Theorem 2.4 and Proposition 2.8 hold and thus there is a branch of nontrivial solutions

$$(n_{\varepsilon}, u_{\varepsilon}) \in \mathbb{R}^+ \times (L_p(J, W^2_{p,\mathcal{B}}(\Omega)) \cap W^1_p(J, L_p(\Omega))), \quad 0 < |\varepsilon| < \varepsilon_0,$$

to the problem

$$\begin{aligned} \partial_a u + \mathcal{A}(U, x)u + \mu(u, a)u &= 0, \quad a \in J, \quad x \in \Omega, \\ u(0, x) &= n \int_0^{a_m} b(u, a)u(a, x) \, \mathrm{d}a, \quad x \in \Omega, \\ \mathcal{B}(x)u(a, x) &= 0, \quad a > 0, \quad x \in \partial\Omega, \\ U(x) &= \int_0^{a_m} u(a, x) \, \mathrm{d}a, \quad x \in \Omega, \end{aligned}$$

bifurcating from (n, u) = (1, 0), such that  $u_{\varepsilon}$  is positive for  $\varepsilon > 0$  sufficiently small. If  $\lambda_0(u)$  denotes the largest eigenvalue of  $-A(u) \in \mathcal{L}(E_1, E_0)$  for  $u \in \Sigma$  and if

$$\int_{0}^{a_{m}} b(u,a) e^{-\int_{0}^{a} \mu(u,r) dr} e^{a\lambda_{0}(u)} da \leq 1, \quad u \in \Sigma,$$
(3.17)

we claim that

$$r(Q_u) \le 1$$
,  $u \in \Sigma$ 

Indeed, if  $B_u$  is a positive eigenfunction corresponding to the eigenvalue  $\lambda_0(u)$  of -A(u), then we have  $e^{-aA(u)}B_u = e^{\lambda(u)a}B_u$  for  $a \ge 0$ , hence

$$Q_u B_u = \int_0^{a_m} b(u, a) e^{-\int_0^a \mu(u, r) dr} e^{\lambda_0(u)a} da B_u ,$$

from which we conclude that  $r(Q_u)$  equals the left hand side of (3.17) in view of (2.27). Recalling (2.29) we deduce that bifurcation must be supercritical provided (3.17) holds; that is, for  $\varepsilon \ge 0$  small we have  $n_{\varepsilon} \ge 1$  and  $u_{\varepsilon}$  is nonnegative. Note that  $\lambda_0(u) \le 0$  if  $a_0(u, \cdot) \ge 0$  and  $a_0(u, \cdot) - \operatorname{div}(a_1(u, \cdot)) \ge 0$  in  $\Omega$ ,  $\nu_0 \ge 0$  and  $a_1(u, \cdot) \cdot \nu \ge 0$  on  $\Gamma_1$  (see [1, Rem.11.3]) in which case the term  $e^{\lambda_0(u)a}$  in (3.17) can be neglected. Moreover,  $s(-A(u)) \le 0$  in this case by definition of the spectral bound. If the functions a,  $a_1$ ,  $a_0$  as well as  $\mu$  and b are symmetric with respect to u, that is, if  $a(u, \cdot) = a(-u, \cdot)$  etc., then Proposition 2.10 entails that there is a positive equilibrium solution for any value of  $n_{\varepsilon}, -\varepsilon_0 < \varepsilon < \varepsilon_0$ .

3.4. **Example.** We give some more details on the example presented in the introduction. Let  $a_m \in (0, \infty)$  and let  $\Omega \subset \mathbb{R}^N$ ,  $N \ge 1$ , be a bounded and smooth domain. For  $p \in (N + 2, \infty)$  put

$$E_1 := W_{p,\mathcal{B}}^2(\Omega) := \{ v \in W_p^2(\Omega) ; \partial_\nu v = 0 \} \hookrightarrow E_0 := L_p(\Omega) ,$$
$$\mathbb{E}_1 := L_p(J, W_{p,\mathcal{B}}^2(\Omega)) \cap W_p^1(J, L_p(\Omega)) .$$

Let  $a \in C^3(\mathbb{R})$  satisfy the ellipticity condition  $a(z) \ge \underline{a} > 0, z \in \mathbb{R}$ . Given  $u \in \mathbb{E}_1 \hookrightarrow L_1(J, W^2_{p, \mathcal{B}}(\Omega))$ set  $U := \int_0^{a_m} u(a, \cdot) da \in W^2_{p, \mathcal{B}}(\Omega)$  and define

$$A(u)w := -\nabla_x \cdot \left( \mathfrak{a}(U)\nabla_x w \right), \quad w \in E_1 = W_{p,\mathcal{B}}^2(\Omega),$$

so that (1.12)-(1.15) can be considered as equations in  $E_0 = L_p(\Omega)$  for  $u: J \to L_p(\Omega)$ :

$$\partial_a u + A(u)u + \mu(u, a)u = 0, \quad a \in (0, a_m),$$
  
$$u(0) = n \int_0^{a_m} b(u, a)u(a) \, \mathrm{d}a.$$
 (3.18)

Observe that

$$W_{p,\mathcal{B}}^2(\Omega) \hookrightarrow E_{\varsigma} := \left( L_p(\Omega), W_{p,\mathcal{B}}^2 \right)_{1-1/p,p} \doteq W_{p,\mathcal{B}}^{2(1-1/p)} \hookrightarrow C^1(\bar{\Omega}) , \quad p > N+2 , \tag{3.19}$$

where the interpolation result follows from [28, 4.3.3.Thm.]. Thus  $A \in C^1(\mathbb{E}_1, \mathcal{L}(W^2_{p,\mathcal{B}}(\Omega), L_p(\Omega)))$ . Furthermore, for each  $u \in \mathbb{E}_1$ , -A(u) generates a positive analytic semigroup  $\{e^{-aA(u)}; a \ge 0\}$  of contractions on  $E_0 = L_p(\Omega)$  (see [1, Sect.7,Thm.11.1]), whence (3.3) holds. Note that the spectral bound s(-A(u)) of -A(u) equals 0 since constants are eigenfunctions owing to the Neumann boundary conditions. Let  $\Sigma$  be some open ball in  $\mathbb{E}_1$  around 0. Suppose  $\mu$  satisfies (3.6) and  $\mu_0 := \mu(0)$  satisfies (3.7) with  $\omega_0 = 0$  and (3.8). Further, let  $b \in C^m(\Sigma, L^+_{p'}(J))$  with  $b_0 := b(0) \neq 0$  be normalized such that

$$\int_{0}^{a_{m}} b_{0}(a) e^{-\int_{0}^{a} \mu_{0}(r) \mathrm{d}r} \,\mathrm{d}a = 1 \,. \tag{3.20}$$

Observe that

$$\mathbb{A}_0 := A_0 + \mu_0 := A(0) + \mu(0, \cdot)$$

is a special case of the operators considered in Examples 3.1 and 3.2 and so has the properties derived there. Thus (2.17) holds due to Lemma 2.2 as shown in Example 3.1. Further, (2.4), (2.18), and (2.19) are satisfied

as noted just previous to (2.25). Next, for B := 1, we have  $e^{-aA_0}B = B$ ,  $a \ge 0$ , and thus  $Q_0B = B$  by (3.20), where

$$Q_0 := \int_0^{a_m} b_0(a) \, e^{-\int_0^a \mu_0(r) \mathrm{d}r} \, e^{-aA_0} \, \mathrm{d}a \in \mathcal{K}_+(E_\varsigma) \, .$$

Moreover,  $Q_0$  is a strongly positive compact operator (see Example 3.1) and  $int(E_{\varsigma}^+) \neq \emptyset$  due to (3.19). Since  $r(Q_0)$  is the only eigenvalue of  $Q_0$  with positive eigenfunction in the (nonempty) interior of the positive cone  $E_{\varsigma}^+$  according to [11, Thm.12.3], we conclude  $r(Q_0) = 1$ . Therefore, we are now in a position to apply Theorem 2.4 and Proposition 2.8 and conclude that problem (3.18) admits of a branch of nontrivial solutions

$$(n_{\varepsilon}, u_{\varepsilon}) \in \mathbb{R}^+ \times (L_p(J, W^2_{p, \mathcal{B}}(\Omega)) \cap W^1_p(J, L_p(\Omega))), \quad 0 < |\varepsilon| < \varepsilon_0,$$

of the form

$$u_{\varepsilon}(a,\cdot) = \varepsilon \left( e^{-\int_0^a \mu_0(r) \mathrm{d}r} e^{-aA_0} B + z_{\varepsilon} \right), \quad z_{\varepsilon} \in L_p(J, W^2_{p,\mathcal{B}}(\Omega)) \cap W^1_p(J, L_p(\Omega)),$$

bifurcating from (n, u) = (1, 0) such that  $u_{\varepsilon}(a) \in L_p^+(\Omega)$  for  $a \in [0, a_m)$  and  $\varepsilon > 0$  small.

However, if (n, u) is any positive solution to problem (3.18), then the relation  $u(0) = nQ_u u(0)$  must hold, where

$$Q_u := \int_0^{a_m} b(u, a) \, e^{-\int_0^a \mu(u, r) \mathrm{d}r} \, e^{-aA(u)} \, \mathrm{d}a$$

Therefore, owing to the fact that dispersal alone does not alter the number of individuals, i.e.

$$\int_{\Omega} e^{-aA(u)} \phi \, \mathrm{d}x = \int_{\Omega} \phi \, \mathrm{d}x \,, \quad \phi \in L_p(\Omega) \,, \tag{3.21}$$

it follows by integrating the relation  $u(0) = nQ_u u(0)$  that necessarily

$$1 = n \int_0^{a_m} b(u, a) e^{-\int_0^a \mu(u, r) dr} da =: n q(u) , \qquad (3.22)$$

for any positive solution (n, u) to (3.18), which is the same constraint as in the non-diffusive case (see [8]). This allows us to say more about the direction of bifurcation in a particular situation. Indeed, assume further that

$$b(u,a) \le b(0,a) = b_0(a), \quad \mu(u,a) \ge \mu(0,a) = \mu_0(a)$$
(3.23)

for  $a \in J$  and  $u \in \mathbb{E}_1^+$ , which is a common modeling assumption stating that effects of population densities do neither increase fertility nor decrease mortality. Then  $q(u_{\varepsilon}) \leq q(0) = 1$  in view of (3.20) for the positive solution  $(n_{\varepsilon}, u_{\varepsilon}), \varepsilon > 0$  small, provided by Theorem 3.1. Thus (3.22) entails  $n_{\varepsilon} \geq 1$  for  $\varepsilon > 0$ small, that is, bifurcation must be supercritical, and there is no equilibrium solution other than the trivial  $u \equiv 0$  corresponding to a parameter value n < 1.

We shall point out that the present example simply reflects the non-diffusive case in the sense that our results here could actually be derived from the case  $A \equiv 0$  (see [8]). For this it is enough to observe that  $\lambda_0 = 0$  is an eigenvalue of -A(u) with corresponding constant eigenfunctions.

Moreover, taking B = 1 we have  $\Pi_0(a, 0)B = e^{-\int_0^a \mu_0(s)ds}$  and since the projection onto N in Proposition 2.6 is given by

$$1 - P_M = \left[ w \mapsto \frac{1}{|\Omega|} \int_{\Omega} w(x) \, \mathrm{d}x \right]$$

the direction of bifurcation, given by  $\zeta$  in Proposition 2.6, can in principle be computed explicitly using (3.21) also if one does not assume (3.23).

**Remark 3.5.** In all our examples we omitted a dependence of  $\mu$  and b on the spatial variable x for simplicity. It is clear though that such a dependence can be included as well. Moreover, we omitted an explicit dependence of the diffusion operator on the age variable. However, the results of Section 2 clearly apply to operators of the form A = A(u, a) as well provided the dependence on a is suitable, e.g. Hölder continuous, see Remarks 2.3(a).

### REFERENCES

- [1] H. Amann. Dual semigroups and second order linear elliptic boundary value problems. Israel J. Math. 45 (1983), 225-254.
- [2] H. Amann. Linear and Quasilinear Parabolic Problems, Volume I: Abstract Linear Theory. Birkhäuser, Basel, Boston, Berlin 1995.
- [3] H. Amann, J. Escher. Analysis II. Birkhäuser, Basel 1999.
- [4] S. Busenberg, M. Iannelli. A system of nonlinear degenerate parabolic equations. J. Reine Angew. Math. 371 (1986), 1-15.
- [5] S. Busenberg, M. Langlais. Global behaviour in age structured S.I.S. models with seasonal periodicities and vertical transmission. J. Math. Anal. Appl. 213 (1997), 511-533.
- [6] Ph. Clément, H.J.A.M. Heijmans et al. One-parameter semigroups. North Holland, CWI Monograph 5, Amsterdam 1987,
- [7] J.M. Cushing. Nontrivial periodic solutions of integrodifferential equations. J. Integral Equations 1 (1979), 165-181.
- [8] J.M. Cushing. Existence and stability of equilibria in age-structured population dynamics. J. Math. Biology 20 (1984), 259-276.
- [9] J.M. Cushing. Global branches of equilibrium solutions of the McKendrick equations for age structured population growth. Comp. Math. Appl. 11 (1985), 175-188.
- [10] J. Cushing. Equilibria in structured populations. J. Math. Biology 23 (1985), 15-39.
- [11] D. Daners, P. Koch-Medina. Abstract Evolution Equations, Periodic Problems, and Applications. Pitman Res. Notes Math. Ser., 279, Longman, Harlow 1992.
- [12] M. Delgado, M. Molina-Becerra, A. Suárez. A nonlinear age-dependent model with spatial diffusion. J. Math. Anal. Appl. 313 (2006), 366-380.
- [13] M. Delgado, M. Molina-Becerra, A. Suárez. Nonlinear age-dependent diffusive equations: A bifurcation approach. J. Diff. Equations 244 (2008), 2133-2155.
- [14] G. Fragnelli, A. Idrissi, L. Maniar. The asymptotic behavior of a population equation with diffusion and delayed birth process. Discrete Contin. Dyn. Syst. Ser. B 7 (2007), 735-754.
- [15] M. Iannelli, M. Martcheva, F.A. Milner. Gender structured population modeling. SIAM Frontiers in Applied Mathematics. Philadelphia 2005.
- [16] M. Langlais. Large time behavior in a nonlinear age-dependent population dynamics problem with spatial diffusion. J. Math. Biol. 26 (1988), 319-346.
- [17] M. Langlais. A nonlinear problem in age-dependent population diffusion. SIAM J. Math. Anal. 16 (1985), 510-529.
- [18] Ph. Laurençot, Ch. Walker. Proteus mirabilis swarm-colony development with drift. To appear in: J. Math. Pures Appl.
- [19] P. Magal, H. Thieme. Eventual compactness for semiflows generated by nonlinear age-structured models. Commun. Pure Appl. Anal. 3, No. 4 (2004), 695-727.
- [20] J. Prüß. Equilibrium solutions of age-specific population dynamics of several species. J. Math. Biol. 11 (1981), 65-84.
- [21] J. Prüß. On the qualitative behaviour of populations with age-specific interactions. Comput. Math. Appl. 9 (1983), 327-339.
- [22] J. Prüß. Stability analysis for equilibria in age-specific population dynamics. Nonl. Anal. Th. Math. Appl. 7 (1983), 1291-1313.
- [23] A. Rhandi, R. Schnaubelt. Asymptotic behaviour of a non-autonomous population equation with diffusion in L<sup>1</sup>. Discrete Contin. Dyn. Syst. 5 (1999), 663-683.
- [24] W. Rudin. Functional Analysis. International Series in Pure and Applied Mathematics. 2<sup>nd</sup> edition. McGraw-Hill Inc. 1991.
- [25] J. Saal. Maximal regularity for the Stokes system on noncylindrical space-time domains. J. Math. Soc. Japan 58 (2006), 617-641.
- [26] H.H. Schaefer. Topological Vector Spaces. Graduate Texts in Mathematics. Springer Verlag, New York, Heidelberg, Berlin 1971.
- [27] H. Thieme. Positive perturbation of operator semigroups: growth bounds, essential compactness, and asynchronous exponential growth. Discrete Contin. Dyn. Syst. 4 (1998), 735-764.
- [28] H. Triebel. Interpolation theory, function spaces, differential operators. Second edition. Johann Ambrosius Barth. Heidelberg, Leipzig 1995.
- [29] Ch. Walker. Global existence for an age and spatially structured haptotaxis model with nonlinear age-boundary conditions. Europ. J. Appl. Math. **19** (2008), 113-147.
- [30] Ch. Walker. Age-dependent equations with nonlinear diffusion. To appear in: Discrete Contin. Dyn. Syst. A.
- [31] Ch. Walker. Global bifurcation of positive equilibria in nonlinear population models. Preprint (2009).
- [32] G.F. Webb. Theory of nonlinear age-dependent population dynamics. Marcel Dekker, New York 1985.
- [33] G.F. Webb. Population models structured by age, size, and spatial position. In: P. Magal, S. Ruan (eds.) Structured Population Models in Biology and Epidemiology. Lecture Notes in Mathematics, Vol. 1936. Springer, Berlin, 2008.

LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY.

E-mail address: walker@ifam.uni-hannover.de