STEADY STATES FOR A COAGULATION-FRAGMENTATION EQUATION WITH VOLUME SCATTERING

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ABSTRACT. A coagulation-fragmentation equation including volume scattering and collisional breakage is considered. We prove that the equation admits steady states of arbitrary mass provided that the kernels satisfy some suitable growth conditions. On the other hand, we also show that zero is the only steady state in particular cases.

1. INTRODUCTION

The aim of this paper is to investigate the existence of steady states to a coagulation-fragmentation equation including a volume scattering effect. Recall that coagulation-fragmentation models describe the time evolution of a system consisting of a very large number of particles, which can either coalesce to form larger particles or split into smaller ones. Usually, these particles are supposed to be identified by their size (mass, volume) only, which, in the conventional continuous models, might be any positive real number. The model considered in the present paper pays attention to the obvious fact that there are no arbitrarily large particles in nature, i.e., a maximal particle size $y_0 \in (0,\infty)$ is introduced beyond which no particle can survive. This feature requires an additional mechanism, called *scattering* in the sequel, preventing the occurrence of particles of size larger than the maximal size y_0 [6]. Besides this scattering phenomenon, the subsequent model also includes the possibility of collisional breakage.

Denoting by $f(t,y) \ge 0$ the density of particles of size $y \in Y := (0,y_0)$ at time $t \ge 0$ (per unit volume), the evolution of the system of particles undergoing simultaneously coagulation and fragmentation can be described by the equation

$$\begin{aligned} \partial_t f &= L(f) \ , \quad (t,y) \in (0,\infty) \times Y \ , \\ f(0,y) &= f_0(y) \ , \quad y \in Y \ , \end{aligned} \tag{1.1}$$

where f_0 is a given initial distribution. The reaction terms $L(f) := L_{\mathsf{b}}(f) + L_{\mathsf{c}}(f) + L_{\mathsf{s}}(f)$ are defined by

$$\begin{aligned} L_{\mathsf{b}}(f)(y) &:= \int_y^{y_0} \gamma(y',y)\ f(y')\ \mathrm{d}y' - f(y) \int_0^y \frac{y'}{y}\ \gamma(y,y')\ \mathrm{d}y' \ , \\ L_{\mathsf{c}}(f)(y) &:= \frac{1}{2} \int_0^y K(y',y-y')\ P(y',y-y')\ f(y-y')\ f(y')\ \mathrm{d}y' \\ &\quad + \frac{1}{2} \int_y^{y_0} \int_0^{y'} K(y'',y'-y'')\ Q(y'',y'-y'')\ \beta_{\mathsf{c}}(y',y)\ f(y'')\ f(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\ &\quad - f(y) \int_0^{y_0-y} K(y,y')\ \{P(y,y')+Q(y,y')\}\ f(y')\ \mathrm{d}y' \ , \\ L_{\mathsf{s}}(f)(y) &:= \frac{1}{2} \int_{y_0}^{2y_0} \int_{y'-y_0}^{y_0} K(y'',y'-y'')\ \beta_{\mathsf{s}}(y',y)\ f(y'')\ f(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\ &\quad - f(y) \int_{y_0-y}^{y_0} K(y,y')\ f(y')\ \mathrm{d}y' \ , \end{aligned}$$

for $y \in Y$, and describe the following reactions:

- The linear operator $L_{\mathsf{b}}(f)$ accounts for the gain and loss of particles of size y due to multiple spontaneous breakage, where $\gamma(y,y') \geq 0$ denotes the rate at which a particle of size $y \in Y$ decays into a particle of size $y' \in (0,y)$.
- Furthermore, two particles y and y' with cumulative size $y + y' < y_0$ can collide at a rate $K(y,y') \geq 0$ and either nothing happens — meaning that the involved particles remain unchanged, for instance, in case of grazing particles — or then they merge with probability $P(y,y')$, or, in case of high-energy collisions, shatter with probability $Q(y,y')$ into several particles according to the shattering distribution $\beta_{\mathsf{c}}(y+y',y'')$ (the latter process is also referred to as collisional breakage). Consistency of the model then demands
$$0 \leq P(y,y') + Q(y,y') \leq 1\ , \quad y + y' < y_0\ . \tag{1.2}$$
These processes are reflected by the operator $L_{\mathsf{c}}(f)$.
- Finally, the scattering operator $L_{\mathsf{s}}(f)$ represents the interaction of two particles y and y' with cumulative size beyond the maximal size y_0. They can coalesce but the resulting particle instantaneously splits into particles all with size within the admissible range Y. The daughter particles are then distributed according to $\beta_{\mathsf{s}}(y+y',y'') \geq 0$. We refer to [6], where the volume scattering mechanism has been introduced for the first time (see also [5] for a more detailed discussion on the modelling issue).

Since there is no particle in- nor outlet, one intuitively expects the total mass to be preserved during time, i.e.
$$\int_0^{y_0} y\ f(t,y)\ \mathrm{d}y = \int_0^{y_0} y\ f_0(y)\ \mathrm{d}y\ , \quad t \geq 0\ . \tag{1.3}$$
Provided that the shattering and the scattering processes are mass preserving (see assumptions (1.10) and (1.12)), this is indeed the case.

From a mathematical viewpoint, some properties of the coagulation-fragmentation equation with volume scattering (1.1) have been investigated recently: in particular, results concerning the well-posedness of (1.1) are to be found in [3, 6, 14, 15] while results on the large time behaviour of the solutions in some cases have been obtained in [16]. As for numerical simulations, we refer to [12]. We also mention at this point that, formally, the classical coagulation-fragmentation model usually contemplated in the literature can be derived from (1.1) by putting $y_0 := \infty$ and $P \equiv 1$ (implying $Q \equiv 0$ according to (1.2)). In particular, the shattering and scattering terms vanish in this case. A survey of the present state of knowledge on the classical coagulation-fragmentation equations and references to further literature for this case can be found in [2, 11].

In the present paper, we will focus on existence of non-trivial steady states to (1.1), that is, on non-zero solutions to the equation
$$L(f) = 0 \ \text{ in } \ Y\ . \tag{1.4}$$
Addressing this issue is mainly motivated by the study of the asymptotic behaviour of solutions to (1.1). Note that the equality (1.3) entails a natural side condition, namely to solve (1.4) subject to
$$\varrho = \int_0^{y_0} y\ f(y)\ \mathrm{d}y\ , \tag{1.5}$$
where $\varrho > 0$ is a given positive real number. Let us point out right now that (1.4), (1.5) does not always possess solutions for $\varrho > 0$. In particular, if $\gamma \equiv 0$ and merely binary shattering and binary scattering are taken into account, zero is the only steady state (see section 4). So far, existence of solutions to (1.4) apart from zero is known only if the kernels satisfy an extended version of the so-called detailed balance condition [15, 16], namely that there exists $H \in L^1(Y)$ such that
$$\gamma(y+y',y)\ H(y+y') \ = \ P(y,y')\ K(y,y')\ H(y)\ H(y')$$

for $0 < y + y' < y_0$, that

$$\beta_c(y,y')\ Q(y'',y-y'')\ K(y'',y-y'')\ H(y'')\ H(y-y'') = \beta_c(y,y'')\ Q(y',y-y')\ K(y',y-y')\ H(y')\ H(y-y')$$

for $0 < y + y'\ ,\ y + y'' < y_0$ and that

$$\beta_s(y,y')\ K(y'',y-y'')\ H(y'')\ H(y-y'')\ =\ \beta_s(y,y'')\ K(y',y-y')\ H(y')\ H(y-y')$$

for $0 < y - y_0 < y', y'' < y_0$. In this case, each function $f_\alpha(y) := H(y)\,\alpha^y$, $y \in Y$, with $\alpha \geq 0$ satisfies (1.4). For the classical coagulation-fragmentation equation, that is, equation (1.1) with $y_0 = \infty$ and $P \equiv 1$, this condition has previously been used in various papers (for instance, see [1, 7, 9, 10]) and the long time behaviour of solutions has been investigated. That the latter equation admits non-trivial and smooth steady state solutions of arbitrary mass without assuming the detailed balance condition has been proven in [8] for constant fragmentation kernels γ and coagulation kernels

$$K(y,y') = a + b(y+y')\ ,\quad y,y' > 0\ ,$$

with $a, b \geq 0$. More recently, existence of non-trivial stationary solutions (in a weak sense) is shown for kernels of the form

$$\gamma(y,y') = y^\sigma B(y'/y)\ ,\quad K(y,y') = y^\alpha (y')^\nu + y^\nu (y')^\alpha$$

with $-1 \leq \alpha \leq 0 \leq \nu \leq 1$, $\alpha + \nu \in [0,1)$, $\sigma \geq -1$, and some suitable function B [4]. To the best of our knowledge, these are the only available results on existence of steady states for the classical coagulation-fragmentation equation ($y_0 = \infty$, $P \equiv 1$) in the absence of the detailed balance condition.

As for (1.4), (1.5), no result seems to be known but the existence of steady states is strongly supported by the numerical simulations in [12]. In this paper, we identify a class of data $(\gamma, K, \beta_c, \beta_s)$ for which (1.4), (1.5) has at least one solution for every $\varrho > 0$. Before stating precisely our assumptions, let us first outline the approach we employ to solve (1.4), (1.5): in such a situation, a natural tool is the Schauder fixed point theorem, but its application requires some strong compactness which is not likely to be available here. Indeed, the operator L is an integral operator and does not seem to be compact. To overcome this difficulty, we consider the regularised problem $-\varepsilon\ f'' = L(f)$ in Y with suitable boundary conditions, where f'' denotes the second derivative of f and $\varepsilon \in (0,1)$. It is then possible to use the Schauder fixed point theorem to establish the existence of a solution f_ε to this problem which satisfies (1.5). The next step is to show that (f_ε) is relatively weakly sequentially compact in $L^1(Y)$ and that its cluster points for this topology solve (1.4), (1.5). Let us further mention that another way to remedy to the lack of strong compactness properties of L has been developed in [4] and relies on the Tychonov fixed point theorem which only demands weak compactness.

The assumptions made throughout this paper are as follows. We suppose that the coagulation kernel K belongs to $L^\infty(Y \times Y)$ and satisfies

$$K_\star (yy')^\sigma \leq K(y,y') = K(y',y) \leq K^\star (yy')^\sigma\ ,\quad (y,y') \in Y \times Y\ , \tag{1.6}$$

for some $K_\star, K^\star > 0$, $\sigma \in [0,1]$ and the monotonicity condition

$$K(y-y',y') \leq K(y,y') \quad \text{for}\ \ 0 < y' < y < y_0\ . \tag{1.7}$$

The probabilities P and Q are non-negative symmetric functions defined on

$$\Xi := \{(y,y') \in Y \times Y\ ;\ y + y' < y_0\}$$

obeying (1.2) and there is $P_\star \in (0,1)$ such that

$$P(y,y') + 2\ Q(y,y') \geq P_\star > 0 \quad \text{for a.a.}\ \ (y,y') \in \Xi\ . \tag{1.8}$$

In addition, P satisfies the monotonicity condition

$$P(y-y',y') \le P(y,y') \quad \text{for a.a.} \quad (y,y') \in \Xi \quad \text{with} \quad 0<y'<y<y_0 \ . \tag{1.9}$$

The fragmentation kernel γ and the shattering distribution β_{c} are non-negative measurable functions defined on

$$\Delta := \{(y,y') \in Y \times Y \, ; \, 0<y'<y<y_0\} \ ,$$

and shattering is supposed to be a mass preserving process, that is,

$$Q(y,y') \left(\int_0^{y+y'} y'' \, \beta_{\mathsf{c}}(y+y',y'') \, \mathrm{d}y'' - y - y' \right) = 0 \quad \text{for a.a.} \quad (y,y') \in \Xi \ . \tag{1.10}$$

We also assume that shattering is suitably dominated by coagulation, i.e., we assume that there exist $z_0 \in Y$ and $\kappa_0 > 0$ with

$$Q(y,y') \int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'') \, \mathrm{d}y'' \ \le \ P(y,y') + 2\, Q(y,y') - \kappa_0 \quad \text{for a.a.} \quad y+y' < z_0 \ . \tag{1.11}$$

The scattering kernel β_{s} is a non-negative measurable function defined on $(y_0, 2y_0) \times (0, y_0)$ and satisfies

$$\int_0^{y_0} y' \, \beta_{\mathsf{s}}(y,y') \, \mathrm{d}y' = y \quad \text{for a.a.} \quad y \in (y_0, 2y_0) \ . \tag{1.12}$$

Finally, we suppose that there are $p > 1$ and μ_γ, μ_{c}, $\mu_{\mathsf{s}} > 0$ such that

$$\int_0^{y} (y')^{\sigma(1-2p)} \, \gamma(y,y')^p \, \mathrm{d}y' \ \le \ \mu_\gamma \quad \text{for a.a.} \quad y \in Y \ , \tag{1.13}$$

$$Q(y,y') \int_0^{y+y'} (y'')^{\sigma(1-p)} \, \beta_{\mathsf{c}}(y+y',y'')^p \, \mathrm{d}y'' \ \le \ \mu_{\mathsf{c}} \quad \text{for a.a.} \quad (y,y') \in \Xi \ , \tag{1.14}$$

$$\int_0^{y_0} (y')^{\sigma(1-p)} \, \beta_{\mathsf{s}}(y,y')^p \, \mathrm{d}y' \ \le \ \mu_{\mathsf{s}} \quad \text{for a.a.} \quad y \in (y_0, 2y_0) \ . \tag{1.15}$$

Note that (1.13)-(1.15) and Hölder's inequality imply

$$\int_0^{y} \gamma(y,y') \, \mathrm{d}y' \ \le \ m_\gamma \, y^\sigma \quad \text{for a.a.} \quad y \in Y \ , \tag{1.16}$$

$$Q(y,y') \int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'') \, \mathrm{d}y'' \ \le \ m_{\mathsf{c}} \quad \text{for a.a.} \quad (y,y') \in \Xi \ , \tag{1.17}$$

$$\int_0^{y_0} \beta_{\mathsf{s}}(y,y') \, \mathrm{d}y' \ \le \ m_{\mathsf{s}} \quad \text{for a.a.} \quad y \in (y_0, 2y_0) \ , \tag{1.18}$$

for some constants $m_\gamma, m_{\mathsf{c}}, m_{\mathsf{s}} > 0$.

Possible (and reasonable) choices of kernels obeying all of the assumptions above are as follows: suppose that K is of the form

$$K(y,y') := A + B(yy')^\theta + C(y+y')^\mu$$

with $A > 0$ and $B, C, \theta, \mu \ge 0$. Let P and Q be non-negative functions such that, for some $\tau, q > 0$,

$$Q(y,y') := q(y+y)^\tau \ , \quad y+y' < y_0 \ ,$$

and such that (1.2), (1.8) and (1.9) hold. For $\bar{\gamma}, \alpha > 0$ and $0 \ge \zeta, \xi, \nu > -1$ define

$$\begin{aligned} \gamma(y,y') &:= \bar{\gamma} \, y^{\alpha-\zeta-1} \, (y')^\zeta \ , \\ \beta_{\mathsf{c}}(y,y') &:= (\xi+2) \, y^{-1-\xi} \, (y')^\xi \ , \\ \beta_{\mathsf{s}}(y,y') &:= (\nu+2) \, y_0^{-2-\nu} \, y \, (y')^\nu \ . \end{aligned}$$

We may then choose $p \in (1, 1+\tau)$ with $-1/p < \min\{\nu, \xi, \zeta, \alpha - 1\}$ so that all assumptions are satisfied with $\sigma := 0$.
On the other hand, if $K(y, y') := \tilde{K}(yy')^\sigma$, $\sigma \in (0, 1]$, $\tilde{K} > 0$, and if $P, Q, \gamma, \beta_{\mathsf{c}}, \beta_{\mathsf{s}}$ are as above with additionally $\alpha > \sigma$ and $1 + \zeta > \sigma$, we find again $p > 1$ small such that (1.13)-(1.15) hold.

Our main result then reads:

Theorem 1.1. *Let (1.6)-(1.15) hold. Then, given any $\varrho > 0$, there exists a non-negative function $f \in L^1_+(Y, y^\sigma \mathrm{d}y)$ satisfying $L(f) = 0$ a.e. in Y and*

$$\int_0^{y_0} y\ f(y)\ \mathrm{d}y = \varrho\ ,$$

where $L^1_+(Y, y^\sigma \mathrm{d}y)$ denotes the positive cone of $L^1(Y, y^\sigma \mathrm{d}y)$.

Recall that the positive cone $L^1_+(Y, y^\sigma \mathrm{d}y)$ of $L^1(Y, y^\sigma \mathrm{d}y)$ is the set of functions of $L^1(Y, y^\sigma \mathrm{d}y)$ which are non-negative almost everywhere in Y.

The solution f to (1.4), (1.5) we construct in Theorem 1.1 actually belongs to $L^p(Y, y^\sigma \mathrm{d}y)$, but it is yet unclear whether f enjoys additional regularity properties. Also, uniqueness of a solution to (1.4), (1.5) does not seem to be obvious.

As already mentioned, the main idea in order to prove this theorem is to consider first a parameter-dependent regularised problem, which can be solved by a Schauder fixed point argument, and to show afterwards that the family of solutions is weakly compact in $L^p(Y, y^\sigma \mathrm{d}y)$. This then guarantees the existence of non-trivial steady states. In the next section we state and solve the regularised problem. Subsequently, we derive in section 3 some uniform estimates leading to the desired weak compactness. In the concluding section 4 we show that problem (1.4), (1.5) does not necessarily have a solution.

2. A regularised problem: existence

Note that (1.6), (1.16), (1.17) and (1.18) imply, for $f \in L^1(Y, y^\sigma\, dy)$, that the reaction terms $L_{\mathsf{b}}(f)$, $L_{\mathsf{c}}(f)$ and $L_{\mathsf{s}}(f)$ belong to $L^1(Y)$. In addition, given $\psi \in L^\infty(Y)$, we have the identities (see [14, Lemma 2.6] or [15, Lemma 2.7])

$$\int_0^{y_0} \psi(y)\ L_{\mathsf{b}}(f)(y)\ \mathrm{d}y\ =\ \int_0^{y_0} \int_0^{y} \left[\psi(y') - \frac{y'}{y}\ \psi(y)\right]\ \gamma(y, y')\ \mathrm{d}y'\ f(y)\ \mathrm{d}y\ , \tag{2.1}$$

$$\int_0^{y_0} \psi(y)\ L_{\mathsf{c}}(f)(y)\ \mathrm{d}y\ =\ \frac{1}{2} \int_0^{y_0} \int_0^{y_0 - y} \psi_{\mathsf{c}}(y, y')\ K(y, y')\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y\ , \tag{2.2}$$

$$\int_0^{y_0} \psi(y)\ L_{\mathsf{s}}(f)(y)\ \mathrm{d}y\ =\ \frac{1}{2} \int_0^{y_0} \int_{y_0 - y}^{y_0} \psi_{\mathsf{s}}(y, y')\ K(y, y')\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y\ , \tag{2.3}$$

where

$$\begin{aligned} \psi_{\mathsf{c}}(y, y')\ &:=\ P(y, y')\ \psi(y + y') - [P(y, y') + Q(y, y')]\ [\psi(y) + \psi(y')] \\ &\quad + Q(y, y')\ \int_0^{y+y'} \psi(y'')\ \beta_{\mathsf{c}}(y + y', y'')\ \mathrm{d}y''\ , \\ \psi_{\mathsf{s}}(y, y')\ &:=\ \int_0^{y_0} \psi(y'')\ \beta_{\mathsf{s}}(y + y', y'')\ \mathrm{d}y'' - \psi(y) - \psi(y')\ . \end{aligned}$$

If $f \in L^q(Y; y^k \mathrm{d}y)$ for some $q \in [1, \infty)$ and $k \in \mathbb{R}$, we define

$$M_{k,q}(f) := \int_0^{y_0} y^k\ |f|^q(y)\ \mathrm{d}y \quad \text{and} \quad M_k(f) := M_{k,1}(f)\ .$$

Given $\delta \in (0, 1)$, we set

$$K_\delta(y, y') := K(y, y') + \delta\ ,$$

and notice that

$$\|K_\delta\|_\infty \le \|K\|_\infty + 1 \ .$$

Hereafter, we denote by $L_\delta(f)$ the reaction terms $L(f)$ but with K_δ instead of K. For $\varepsilon \in (0,\delta)$ and $\varrho \in (0,\infty)$ we define

$$\omega^2 := \frac{\|K_\delta\|_\infty}{\varepsilon} + m_\gamma\ y_0^\sigma \quad \text{and} \quad R := \frac{1}{4\,\varepsilon\, y_0}\left(6\ \varepsilon\ \varrho + 5\ \omega^2\ y_0^2\ \varrho + 3\ \|K_\delta\|_\infty\ y_0\ \varrho^2\right) \ . \tag{2.4}$$

We next introduce

$$F(f) := \varphi_\varepsilon(f)\ L_\delta(f) + \omega^2\ f$$

for $f \in L^1_+(Y)$, where

$$\varphi_\varepsilon(f) := \frac{1}{1+\varepsilon\, M_0(f)}\ ,$$

and observe that $F(f)$ belongs to $L^1(Y)$. We then denote by u_f the unique solution in $W^{2,1}(Y)$ to the boundary-value problem

$$-\varepsilon\ u_f'' + \omega^2\ u_f = F(f) \quad \text{in} \quad Y\,, \tag{2.5}$$

$$u_f(0) = y_0\ u_f'(y_0) - u_f(y_0) = 0\ . \tag{2.6}$$

Finally, let $\mathcal{C}$ be the subset of $L^1(Y)$ defined by

$$\mathcal{C} := \left\{ f \in L^1_+(Y)\,;\ M_1(f) = \varrho\,,\ M_0(f) \le R \right\}\ . \tag{2.7}$$

Clearly, $\mathcal{C}$ is a non-empty, bounded and closed convex subset of $L^1(Y)$. In addition, we have the following property:

Lemma 2.1. *If $f \in \mathcal{C}$, then $u_f \in \mathcal{C}$.*

Proof. Since $f \ge 0$, it follows from (1.2), (1.6) and (1.16) that

$$\begin{aligned} F(f)(y) &\ge \omega^2\ f(y) - \varphi_\varepsilon(f)\ f(y)\ \left(\int_0^y \frac{y'}{y}\ \gamma(y,y')\ \mathrm{d}y' + \int_0^{y_0} K_\delta(y,y')\ f(y')\ \mathrm{d}y'\right) \\ &\ge \omega^2\ f(y) - \varphi_\varepsilon(f)\ f(y)\ \left(m_\gamma\ y_0^\sigma + \|K_\delta\|_\infty\ M_0(f)\right) \\ &\ge 0\ . \end{aligned}$$

The comparison principle then entails that $u_f \ge 0$. We next readily infer from (1.10), (1.12) and (2.1)-(2.3) that

$$\int_0^{y_0} y\ F(f)(y)\ \mathrm{d}y = \omega^2\ M_1(f)\ ,$$

while (2.6) yields that

$$-\varepsilon \int_0^{y_0} y\ u_f''(y)\ \mathrm{d}y = 0\,.$$

Consequently, we deduce from (2.5) after multiplication by y and integration over Y the equality $M_1(u_f) = M_1(f)$.

We now multiply (2.5) by y^3 and integrate over Y. Observe that

$$-\int_0^{y_0} y^3\ u_f''(y)\ \mathrm{d}y = 2\ y_0^2\ u_f(y_0) - 6\ M_1(u_f)$$

by (2.6), while (1.2), (1.10), (1.12) and (2.1)-(2.3) yield

$$
\begin{aligned}
&\int_0^{y_0} y^3\ F(f)(y)\ \mathrm{d}y \\
&\quad \le \omega^2\ M_3(f) + \frac{3\ \|K_\delta\|_\infty}{2}\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_0^{y_0-y} \left(y^2\ y' + y\ y'^2\right)\ P(y,y')\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\qquad + \frac{\|K_\delta\|_\infty}{2}\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_0^{y_0-y} \left[(y+y')^2\ \int_0^{y+y'} y''\ \beta_{\mathrm{c}}(y+y',y'')\ \mathrm{d}y'' - y^3 - y'^3\right] \\
&\qquad\qquad \times\ Q(y,y')\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\qquad + \frac{\|K_\delta\|_\infty}{2}\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_{y_0-y}^{y_0} \left[y_0^2\ \int_0^{y_0} y''\ \beta_{\mathrm{s}}(y+y',y'')\ \mathrm{d}y'' - y^3 - y'^3\right]\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\quad \le \omega^2\ y_0^2\ M_1(f) + \frac{3\ \|K_\delta\|_\infty}{2}\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_0^{y_0-y} \left(y^2\ y' + y\ y'^2\right)\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\qquad + \frac{\|K_\delta\|_\infty}{2}\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_{y_0-y}^{y_0} \left[(y+y')^3 - y^3 - y'^3\right]\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\quad \le \omega^2\ y_0^2\ \varrho + 3\ \|K_\delta\|_\infty\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_0^{y_0} y^2\ y'\ f(y)\ f(y')\ \mathrm{d}y'\mathrm{d}y \\
&\quad \le \omega^2\ y_0^2\ \varrho + 3\ \|K_\delta\|_\infty\ y_0\ \varrho^2\ .
\end{aligned}
$$

Therefore, we obtain

$$
2\ \varepsilon\ y_0^2\ u_f(y_0) \le 6\ \varepsilon\ \varrho + \omega^2\ y_0^2\ \varrho + 3\ \|K_\delta\|_\infty\ y_0\ \varrho^2\ . \tag{2.8}
$$

On the other hand, due to

$$
\begin{aligned}
&\int_0^{y_0} y^2\ F(f)(y)\ \mathrm{d}y \\
&\quad \ge -\ \varphi_\varepsilon(f)\ \int_0^{y_0} \int_0^{y} y\ y'\ \gamma(y,y')\ \mathrm{d}y'\ f(y)\ \mathrm{d}y \\
&\qquad -\ \frac{\varphi_\varepsilon(f)}{2}\ \int_0^{y_0} \int_0^{y_0-y} (y^2+y'^2)\ K_\delta(y,y')\ \left\{P(y,y') + Q(y,y')\right\}\ f(y')\ f(y)\ \mathrm{d}y'\mathrm{d}y \\
&\qquad -\ \frac{\varphi_\varepsilon(f)}{2}\ \int_0^{y_0} \int_{y_0-y}^{y_0} (y^2+y'^2)\ K_\delta(y,y')\ f(y')\ f(y)\ \mathrm{d}y'\mathrm{d}y \\
&\quad \ge -\ m_\gamma\ y_0^{1+\sigma}\ M_1(f) - \varrho\ y_0\ \|K_\delta\|_\infty\ \frac{M_0(f)}{1+\varepsilon\, M_0(f)}
\end{aligned}
$$

and

$$
-\int_0^{y_0} y^2\ u_f''(y)\ \mathrm{d}y\ =\ y_0\ u_f(y_0) - 2\ M_0(u_f)\ ,
$$

we deduce from (2.5) that

$$
\varepsilon\ y_0\ u_f(y_0) - 2\ \varepsilon\ M_0(u_f) + \omega^2\ M_2(u_f) \ge -m_\gamma\ y_0^{1+\sigma}\ \varrho - \varrho\ y_0\ \frac{\|K_\delta\|_\infty}{\varepsilon}\ .
$$

Consequently, taking into account (2.8) and the definition of ω, we end up with

$$
2\ \varepsilon\ M_0(u_f) \le \varepsilon\ y_0\ u_f(y_0) + \omega^2\ y_0\ M_1(u_f) + m_\gamma\ y_0^{1+\sigma}\ \varrho + \varrho\ y_0\ \frac{\|K_\delta\|_\infty}{\varepsilon} \le 2\ \varepsilon\ R\ ,
$$

and the proof is complete. □

Proposition 2.2. *There is a function $f \in \mathcal{C} \cap W^{2,1}(Y)$ such that*

$$-\varepsilon\ f'' = \varphi_\varepsilon(f)\ L_\delta(f) \quad in \quad Y\ , \tag{2.9}$$

$$f(0) = y_0\ f'(y_0) - f(y_0) = 0\ . \tag{2.10}$$

Proof. By Lemma 2.1, the mapping $f \longmapsto u_f$ maps $\mathcal{C}$ into itself. In addition, it is clearly a continuous and compact mapping from $\mathcal{C}$ into itself for the norm-topology of $L^1(Y)$. Indeed, we recall that, for $f \in L^1(Y)$, u_f is given by

$$u_f(y) = \left(\lambda - \int_0^y e^{-\bar\omega y'}\ F(f)(y')\ \mathrm{d}y'\right)\ \frac{e^{\bar\omega y}}{2\,\varepsilon\,\bar\omega}\ -\ \left(\lambda - \int_0^y e^{\bar\omega y'}\ F(f)(y')\ \mathrm{d}y'\right)\ \frac{e^{-\bar\omega y}}{2\,\varepsilon\,\bar\omega}$$

for $y \in [0, y_0]$, where $\bar\omega := \omega\ \varepsilon^{-1/2}$ and

$$\lambda := \vartheta\ \int_0^{y_0} e^{-\bar\omega y'}\ F(f)(y')\ \mathrm{d}y'\ +\ (1-\vartheta)\ \int_0^{y_0} e^{\bar\omega y'}\ F(f)(y')\ \mathrm{d}y'\ , \tag{2.11}$$

with

$$\vartheta := \frac{y_0\ \bar\omega - 1}{y_0\ \bar\omega - 1 + (y_0\ \bar\omega + 1)\ e^{-2\bar\omega y_0}}\ . \tag{2.12}$$

In particular, there is a constant Γ depending on y_0 and ε such that

$$\|u_f\|_{W^{1,\infty}(Y)} \le \Gamma\ \|F(f)\|_{L^1(Y)}\ .$$

Since F is a locally Lipschitz continuous map from $L^1(Y)$ into $L^1(Y)$ (see [14, Lemma 2.1]), the claimed continuity and compactness of $f \longmapsto u_f$ follow.

Now, since $\mathcal{C}$ is a non-empty, closed and convex subset of $L^1(Y)$, we are in a position to apply the Schauder fixed point theorem and conclude that there is $f \in \mathcal{C}$ such that $u_f = f$. Proposition 2.2 readily follows. □

3. A regularised problem: uniform estimates

For $\delta \in (0,1)$, $\varepsilon \in (0,\delta)$ and $\varrho > 0$, we denote by $f_{\varepsilon,\delta}$ the solution to (2.9), (2.10) given by Proposition 2.2. In particular, we have

$$\int_0^{y_0} y\ f_{\varepsilon,\delta}(y)\ \mathrm{d}y = \varrho\ . \tag{3.1}$$

The aim of this section is to prove that $(f_{\varepsilon,\delta})$ is weakly sequentially compact first with respect to ε and subsequently with respect to δ. In the following, we denote by C various positive constants which do neither depend on ε nor on δ. Dependence on δ, for instance, will be indicated explicitly by writing $C(\delta)$.

We first proceed as in Lemma 2.1 to bound $f_{\varepsilon,\delta}(y_0)$.

Lemma 3.1. *For $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$, we have*

$$f'_{\varepsilon,\delta}(0) \ge 0 \quad and \quad \varepsilon\ f_{\varepsilon,\delta}(y_0) \le C\ .$$

Proof. Clearly, $f'_{\varepsilon,\delta}(0) \ge 0$ since $f_{\varepsilon,\delta}(0) = 0$ and $f_{\varepsilon,\delta}(y) \ge 0$ for $y \in Y$. We next multiply (2.9) by y^3 and integrate over Y. As in the proof of Lemma 2.1, we use (2.10) and (3.1) to obtain

$$\varepsilon\ \left(2\ y_0^2\ f_{\varepsilon,\delta}(y_0) - 6\ \varrho\right) \le 3\ \|K_\delta\|_\infty\ y_0\ \varrho^2 \le 3\ (\|K\|_\infty + 1)\ y_0\ \varrho^2\ .$$

□

We next estimate the L^1-norm of $f_{\varepsilon,\delta}$ using a different argument than in the previous section.

Lemma 3.2. *For $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$, we have*

$$\delta^{1/2}\ M_0(f_{\varepsilon,\delta}) + M_\sigma(f_{\varepsilon,\delta}) \le C\ . \tag{3.2}$$

Proof. We integrate (2.9) over Y. Since

$$- \varepsilon \int_0^{y_0} f''_{\varepsilon,\delta}(y)\ dy \ = \ - \varepsilon\ \left(f'_{\varepsilon,\delta}(y_0) - f'_{\varepsilon,\delta}(0)\right) \geq -\varepsilon\ \frac{f_{\varepsilon,\delta}(y_0)}{y_0} \geq -C$$

by (2.10) and Lemma 3.1, we deduce from (1.11), (1.16)-(1.18) and (2.1)-(2.3) that

$$\begin{aligned}
-\frac{C}{\varphi_\varepsilon(f_{\varepsilon,\delta})} \ &\leq - \frac{\varepsilon}{\varphi_\varepsilon(f_{\varepsilon,\delta})} \int_0^{y_0} f''_{\varepsilon,\delta}(y)\ dy \\
&\leq \int_0^{y_0} \int_0^{y} \left(1 - \frac{y'}{y}\right)\ \gamma(y,y')\ dy'\ f_{\varepsilon,\delta}(y')\ dy \\
&\quad - \frac{\kappa_0}{2} \int_0^{y_0} \int_0^{y_0-y} \mathbf{1}_{[0,z_0]}(y+y')\ K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy \\
&\quad + \frac{1}{2}\ m_{\mathsf{c}} \int_0^{y_0} \int_0^{y_0-y} \mathbf{1}_{[z_0,y_0]}(y+y')\ K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy \\
&\quad + \frac{1}{2}\ (m_{\mathsf{s}} - 2) \int_0^{y_0} \int_{y_0-y}^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy \\
&\leq m_\gamma\ M_\sigma(f_{\varepsilon,\delta}) - \frac{\kappa_0}{2} \int_0^{y_0} \int_0^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy \\
&\quad + C \int_0^{y_0} \int_0^{y_0} \mathbf{1}_{[z_0,y_0]}(y+y')\ K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy\ .
\end{aligned}$$

Owing to (1.6) and the definition of K_δ, we have

$$\mathbf{1}_{[z_0,y_0]}(y+y')\ K_\delta(y,y') \leq \frac{y+y'}{z_0}\ \left(K^\star (yy')^\sigma + \delta\right)\ , \quad (y,y') \in Y \times Y\ ,$$

and thus, thanks to (3.1),

$$\begin{aligned}
-\frac{C}{\varphi_\varepsilon(f_{\varepsilon,\delta})} \ &\leq m_\gamma\ M_\sigma(f_{\varepsilon,\delta}) - \frac{\kappa_0}{2}\ \left(K_\star\ M_\sigma(f_{\varepsilon,\delta})^2 + \delta\ M_0(f_{\varepsilon,\delta})^2\right) \\
&\quad + C \int_0^{y_0} \int_0^{y_0} y\ \ (K^\star\ (y\ y')^\sigma + \delta)\ \ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y)\ dy'dy \\
&\leq m_\gamma\ M_\sigma(f_{\varepsilon,\delta}) - \frac{\kappa_0}{2}\ \left(K_\star\ M_\sigma(f_{\varepsilon,\delta})^2 + \delta\ M_0(f_{\varepsilon,\delta})^2\right) \\
&\quad + C\ \left(K^\star\ \varrho\ y_0^\sigma\ M_\sigma(f_{\varepsilon,\delta}) + \varrho\ \delta\ M_0(f_{\varepsilon,\delta})\right) \\
&\leq C - \frac{\kappa_0}{4}\ \left(K_\star\ M_\sigma(f_{\varepsilon,\delta})^2 + \delta\ M_0(f_{\varepsilon,\delta})^2\right)
\end{aligned}$$

by the Young inequality. Since $\varepsilon \leq \delta$, a further application of the Young inequality entails that

$$K_\star\ M_\sigma(f_{\varepsilon,\delta})^2 + \delta\ M_0(f_{\varepsilon,\delta})^2 \leq C\ \ (1 + \delta\ M_0(f_{\varepsilon,\delta})) \leq C + \frac{\delta}{2}\ M_0(f_{\varepsilon,\delta})^2\ ,$$

whence (3.2). □

We next turn to the cornerstone of the proof, that is, the weak compactness of $(f_{\varepsilon,\delta})$ with respect to ε. More precisely, the following result is true.

Lemma 3.3. *For $\delta \in (0,1)$ and $\varepsilon \in (0,\delta)$, we have*

$$\int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^p\ dy\ \leq\ C(\delta)\ . \tag{3.3}$$

Proof. Owing to (2.10) and the Hölder inequality, we first notice that

$$\begin{aligned}
\left(f_{\varepsilon,\delta}(y_0)\right)^{(1+p)/2} &= \int_0^{y_0} \frac{\mathrm{d}}{\mathrm{d}y} \left[f_{\varepsilon,\delta}(y)\right]^{(1+p)/2}\ \mathrm{d}y \\
&= \frac{1+p}{2} \int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{1/2} \left(f_{\varepsilon,\delta}(y)\right)^{(p-2)/2} f'_{\varepsilon,\delta}(y)\ \mathrm{d}y \\
&\le \frac{1+p}{2} M_0(f_{\varepsilon,\delta})^{1/2} \left(\int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{p-2} |f'_{\varepsilon,\delta}(y)|^2\ \mathrm{d}y \right)^{1/2}
\end{aligned}$$

and therefore

$$\left(f_{\varepsilon,\delta}(y_0)\right)^{1+p} \le C(\delta) \int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{p-2} |f'_{\varepsilon,\delta}(y)|^2\ \mathrm{d}y \tag{3.4}$$

by Lemma 3.2. We now multiply (2.9) by $p\left(f_{\varepsilon,\delta}(y)\right)^{p-1}$ and integrate over Y. From (2.10), (3.4) and the Young inequality we infer that

$$\begin{aligned}
-\varepsilon\, p \int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{p-1} f''_{\varepsilon,\delta}(y)\ \mathrm{d}y &= -\,\varepsilon\, p\, \left(f_{\varepsilon,\delta}(y_0)\right)^{p-1} f'_{\varepsilon,\delta}(y_0) \\
&\quad + \varepsilon\, p\, (p-1) \int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{p-2} |f'_{\varepsilon,\delta}(y)|^2\ \mathrm{d}y \\
&\ge -\,\frac{\varepsilon\, p}{y_0} \left(f_{\varepsilon,\delta}(y_0)\right)^{p} + \varepsilon\, C(\delta) \left(f_{\varepsilon,\delta}(y_0)\right)^{p+1} \\
&\ge -\, C(\delta)\ .
\end{aligned}$$

Consequently,

$$-\,C(\delta) \le -\,\varepsilon\, p \int_0^{y_0} \left(f_{\varepsilon,\delta}(y)\right)^{p-1} f''_{\varepsilon,\delta}(y)\ \mathrm{d}y \le \varphi_\varepsilon(f_{\varepsilon,\delta})\ (I_1+I_2+I_3+I_4-I_5-I_6)\ , \tag{3.5}$$

where we put

$$\begin{aligned}
I_1 &:= \frac{p}{2} \int_0^{y_0} \int_{y_0}^{2y_0} \int_{y'-y_0}^{y_0} K_\delta(y'',y'-y'')\ \beta_{\mathsf{s}}(y',y)\ f_{\varepsilon,\delta}(y'-y'')\ f_{\varepsilon,\delta}(y'')\ \left(f_{\varepsilon,\delta}(y)\right)^{p-1}\ \mathrm{d}y''\mathrm{d}y'\mathrm{d}y\ , \\
I_2 &:= \frac{p}{2} \int_0^{y_0} \int_y^{y_0} \int_0^{y'} \beta_{\mathsf{c}}(y',y)\ K_\delta(y'',y'-y'')\ Q(y'',y'-y'') \\
&\qquad \times f_{\varepsilon,\delta}(y'-y'')\ f_{\varepsilon,\delta}(y'')\ \left(f_{\varepsilon,\delta}(y)\right)^{p-1}\ \mathrm{d}y''\mathrm{d}y'\mathrm{d}y\ , \\
I_3 &:= p \int_0^{y_0} \int_y^{y_0} \gamma(y',y)\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^{p-1}\ \mathrm{d}y'\mathrm{d}y\ , \\
I_4 &:= \frac{p}{2} \int_0^{y_0} \int_0^{y} K_\delta(y',y-y')\ P(y',y-y')\ f_{\varepsilon,\delta}(y-y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^{p-1}\ \mathrm{d}y'\mathrm{d}y\ , \\
I_5 &:= p \int_0^{y_0} \int_0^{y_0-y} K_\delta(y,y')\ \left\{P(y,y')+Q(y,y')\right\}\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^{p}\ \mathrm{d}y'\mathrm{d}y\ , \\
I_6 &:= p \int_0^{y_0} \int_{y_0-y}^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^{p}\ \mathrm{d}y'\mathrm{d}y\ .
\end{aligned}$$

Observe then that (1.15) and the Young inequality imply that, for $\xi \in (0,1)$, there is a constant $C_\xi > 0$ such that

$$\int_0^{y_0} \beta_{\mathsf{s}}(y',y)\ p\ f_{\varepsilon,\delta}(y)^{p-1}\ \mathrm{d}y \le C_\xi\ \mu_{\mathsf{s}} + \xi \int_0^{y_0} y^\sigma \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y = C_\xi\ \mu_{\mathsf{s}} + \xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\ .$$

Therefore, (1.6) and Lemma 3.2 entail that

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0}\int_0^{y_0} \beta_{\mathsf{s}}(y',y)\ p\ \big(f_{\varepsilon,\delta}(y)\big)^{p-1}\ \mathrm{d}y\ K_\delta(y'',y'-y'')\ f_{\varepsilon,\delta}(y'-y'')\ f_{\varepsilon,\delta}(y'')\ \mathrm{d}y''\mathrm{d}y' \\ &\le \big(C_\xi\ \mu_{\mathsf{s}}+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ \int_0^{y_0}\int_{y_0-y'}^{y_0} K_\delta(y'',y')\ f_{\varepsilon,\delta}(y')\ f_{\varepsilon,\delta}(y'')\ \mathrm{d}y''\mathrm{d}y' \\ &\le \big(C_\xi\ \mu_{\mathsf{s}}+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ \left(K^\star\ M_\sigma(f_{\varepsilon,\delta})^2+\frac{2}{y_0}\ \varrho\ \delta\ M_0(f_{\varepsilon,\delta})\right)\ , \end{aligned}$$

$$I_1\ \le\ C\ \big(C_\xi+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ . \tag{3.6}$$

We estimate I_2 analogously and thus obtain from (1.2), (1.6), (1.14) and Lemma 3.2 that, for $\xi\in(0,1)$,

$$\begin{aligned} I_2 &= \frac{1}{2}\int_0^{y_0}\int_0^{y'}\int_0^{y'} \beta_{\mathsf{c}}(y',y)\ p\ \big(f_{\varepsilon,\delta}(y)\big)^{p-1}\ \mathrm{d}y\ Q(y'',y'-y'') \\ &\qquad\qquad \times\ K_\delta(y'',y'-y'')\ f_{\varepsilon,\delta}(y'-y'')\ f_{\varepsilon,\delta}(y'')\ \mathrm{d}y''\mathrm{d}y' \\ &\le \big(C_\xi\ \mu_{\mathsf{c}}+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ \int_0^{y_0}\int_0^{y_0-y} K_\delta(y,y')\ f_{\varepsilon,\delta}(y)\ f_{\varepsilon,\delta}(y')\ \mathrm{d}y'\mathrm{d}y \\ &\le \big(C_\xi\ \mu_{\mathsf{c}}+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ \big(K^\star\ M_\sigma(f_{\varepsilon,\delta})^2+\delta\ M_0(f_{\varepsilon,\delta})^2\big) \end{aligned}$$

$$I_2\ \le\ C\ \big(C_\xi\ +\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ . \tag{3.7}$$

In a similar way, the Young inequality and (1.13) yield

$$\begin{aligned} I_3 &= \int_0^{y_0} f_{\varepsilon,\delta}(y')\ \int_0^{y'} \gamma(y',y)\ p\ \big(f_{\varepsilon,\delta}(y)\big)^{p-1}\ \mathrm{d}y\,\mathrm{d}y' \\ &\le \int_0^{y_0} y'^\sigma\ f_{\varepsilon,\delta}(y')\ \int_0^{y'} p\ y^{-\sigma}\ \gamma(y',y)\ \big(f_{\varepsilon,\delta}(y)\big)^{p-1}\ \mathrm{d}y\,\mathrm{d}y' \\ &\le \int_0^{y_0} y'^\sigma\ f_{\varepsilon,\delta}(y')\ \int_0^{y'} p\ y^{\sigma(1-2p)/p}\ \gamma(y',y)\ \big(y^{\sigma/p}f_{\varepsilon,\delta}(y)\big)^{p-1}\ \mathrm{d}y\,\mathrm{d}y' \\ &\le \int_0^{y_0} y'^\sigma\ f_{\varepsilon,\delta}(y')\ \big(C_\xi\ \mu_\gamma+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ \mathrm{d}y'\ , \end{aligned}$$

whence, by Lemma 3.2,

$$I_3\ \le\ C\ \big(C_\xi+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ M_\sigma(f_{\varepsilon,\delta})\ \le\ C\ \big(C_\xi+\xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\big)\ . \tag{3.8}$$

We now estimate $I_4-I_5-I_6$. For that purpose, observe first that, by the Young inequality,

$$\begin{aligned} I_4 &= \frac{1}{2}\int_0^{y_0}\int_0^{y} K_\delta(y',y-y')\ P(y',y-y')\ f_{\varepsilon,\delta}(y-y')\ f_\delta(y')\ p\ \big(f_{\varepsilon,\delta}(y)\big)^{p-1}\mathrm{d}y'\mathrm{d}y \\ &\le \frac{1}{2}\int_0^{y_0}\int_0^{y_0-y'} K_\delta(y',y)\ P(y',y)\ f_{\varepsilon,\delta}(y)\ \big(f_{\varepsilon,\delta}(y')\big)^p\ \mathrm{d}y\mathrm{d}y' \\ &\quad +\frac{1}{2}\ (p-1)\ \int_0^{y_0}\int_0^{y} K_\delta(y-y',y')\ P(y-y',y')\ f_{\varepsilon,\delta}(y')\ \big(f_{\varepsilon,\delta}(y)\big)^p\ \mathrm{d}y'\mathrm{d}y\ . \end{aligned}$$

Therefore, we derive that

$$\begin{aligned}
I_4 - I_5 - I_6 \ \leq\ & \frac{1}{2}\ \int_0^{y_0} \int_0^{y_0-y} K_\delta(y,y')\ P(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
& + \frac{1}{2}\ (p-1)\ \int_0^{y_0} \int_0^{y} K_\delta(y-y',y')\ P(y-y',y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
& - p\ \int_0^{y_0} \int_0^{y_0-y} K_\delta(y,y')\ (P+Q)(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
& - p\ \int_0^{y_0} \int_{y_0-y}^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
\leq\ & -\ \frac{1}{2}\ p \int_0^{y_0} \int_0^{y_0-y} K_\delta(y,y')\ (P+2\ Q)(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
& -\ \frac{1}{2}\ (p+1) \int_0^{y_0} \int_{y_0-y}^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y \\
\leq\ & -\ \frac{1}{2}\ p\ P_\star\ \int_0^{y_0} \int_0^{y_0} K_\delta(y,y')\ f_{\varepsilon,\delta}(y')\ \left(f_{\varepsilon,\delta}(y)\right)^p\ \mathrm{d}y'\mathrm{d}y\ ,
\end{aligned}$$

where we have used the monotonicity conditions (1.7) and (1.9) to obtain the second inequality and (1.8) for the last inequality (recall that $P_\star \leq 1$). Since $\varrho = M_1(f_{\varepsilon,\delta}) \leq y_0\, M_0(f_{\varepsilon,\delta})$ and $\varrho = M_1(f_{\varepsilon,\delta}) \leq y_0^{1-\sigma}\ M_\sigma(f_{\varepsilon,\delta})$, we deduce from (1.6) that

$$\begin{aligned}
I_4 - I_5 - I_6 \quad & \leq\ -\ \frac{1}{2}\ p\ P_\star\ \left(K_\star\ M_\sigma(f_{\varepsilon,\delta})\ M_{\sigma,p}(f_{\varepsilon,\delta}) + \delta\ M_0(f_{\varepsilon,\delta})\ M_{0,p}(f_{\varepsilon,\delta})\right)\ , \\
I_4 - I_5 - I_6 \quad & \leq\ -\ \frac{1}{2}\ p\ \varrho\ P_\star\ \left(K_\star\ y_0^{\sigma-1}\ M_{\sigma,p}(f_{\varepsilon,\delta}) + \delta\ y_0^{-1}\ M_{0,p}(f_{\varepsilon,\delta})\right)\ . \qquad (3.9)
\end{aligned}$$

Gathering (3.5)-(3.9) we end up with

$$-\ \frac{C(\delta)}{\varphi_\varepsilon(f_{\varepsilon,\delta})}\ \leq\ C\ \left(C_\xi + \xi\ M_{\sigma,p}(f_{\varepsilon,\delta})\right)\ -\ \frac{1}{2\,y_0}\ p\ P_\star\ \varrho\ \delta\ M_{0,p}(f_{\varepsilon,\delta})\ .$$

Choosing then $\xi \in (0,1)$ sufficiently small and noticing that $\left(\varphi_\varepsilon(f_{\varepsilon,\delta})\right)^{-1} \leq (1+\delta\ M_0(f_{\varepsilon,\delta})) \leq C$ due to $\varepsilon \in (0,\delta)$ and Lemma 3.2, the assertion follows since $M_{\sigma,p}(f_{\varepsilon,\delta}) \leq y_0^\sigma\ M_{0,p}(f_{\varepsilon,\delta})$. □

The fact that the monotonicity condition (1.7) on the coagulation kernel K yields L^p-estimates has already been used in [9] for the classical coagulation equation (seee also [11] and the references therein). In addition, the weak compactness in $L^1(Y)$ of the trajectories of (1.1) established in [15] relies on a similar observation. We adapt here this strategy to estimate $I_4 - I_5 - I_6$ under more general assumptions than the one used in [15].

Now the proof of Theorem 1.1 is a consequence of the previous considerations.

Proof of Theorem 1.1. Keeping $\delta \in (0,1)$ fixed, the set $\{f_{\varepsilon,\delta}\,;\,\varepsilon \in (0,\delta)\}$ is bounded in $L^p(Y)$ according to Lemma 3.3. Therefore, there are a sequence $(f_{\varepsilon_n,\delta})$ and $f_\delta \in L^p(Y)$ such that

$$f_{\varepsilon_n,\delta} \rightharpoonup f_\delta \quad \text{in} \quad L^p(Y) \quad \text{as} \quad \varepsilon_n \to 0\ . \qquad (3.10)$$

Since $f_{\varepsilon_n,\delta}$ is non-negative and satisfies $M_1(f_{\varepsilon_n,\delta}) = \varrho$ by (3.1) for each $n \geq 1$, it readily follows from (3.10) that

$$f_\delta \geq 0 \quad \text{a.e. in} \quad Y \quad \text{and} \quad M_1(f_\delta) = \varrho\ . \qquad (3.11)$$

We then claim that $L_\delta(f_\delta) = 0$. Indeed, on the one hand, it is well-known that L_δ is weakly continuous in $L^1(Y)$ (see either the pioneering work [13] or [15, Appendix A] for a complete proof), and the convergence (3.10) ensures that $L_\delta(f_{\varepsilon_n,\delta}) \rightharpoonup L_\delta(f_\delta)$ in $L^1(Y)$. On the other hand, by (3.2), $(-\varepsilon_n f''_{\varepsilon_n,\delta})$ converges to zero in $\mathcal{D}'(Y)$. Consequently,

$$\int_0^{y_0} L_\delta(f_\delta)(y)\ \psi(y)\ \mathrm{d}y = 0 \qquad \text{for each} \quad \psi \in C_0^\infty(Y)\ , \tag{3.12}$$

whence

$$L_\delta(f_\delta) = 0 \quad \text{a.e. in } Y\ . \tag{3.13}$$

We may now test (3.13) with $p\,(f_\delta)^{p-1}$ and obtain

$$0\ =\ p \int_0^{y_0} \left(f_\delta(y)\right)^{p-1}\ L_\delta(f_\delta)(y)\ \mathrm{d}y\ \le\ I_1 + I_2 + I_3 + I_4 - I_5 - I_6\ , \tag{3.14}$$

where the I_k's are defined as in the proof of Lemma 3.3 but with $f_{\varepsilon,\delta}$ replaced by f_δ. Since (3.10) and (3.2) imply

$$\delta^{1/2}\ M_0(f_\delta) \le C \quad \text{and} \quad M_\sigma(f_\delta) \le C\ , \tag{3.15}$$

we can proceed as in the proof of (3.6), (3.7), (3.8) and (3.9) in Lemma 3.3 to deduce that, for $\xi \in (0,1)$,

$$I_1 + I_2 + I_3\ \le\ C\ \left(C_\xi + \xi\ M_{\sigma,p}(f_\delta)\right) \tag{3.16}$$

and

$$I_4 - I_5 - I_6\ \le\ -\frac{1}{2}\ p\ \varrho\ P_\star\ K_\star\ y_0^{\sigma-1}\ M_{\sigma,p}\big(f_{\varepsilon,\delta}\big)\ . \tag{3.17}$$

Combining (3.14), (3.16) and (3.17) and choosing $\xi \in (0,1)$ sufficiently small, we finally obtain that

$$\int_0^{y_0} y^\sigma\ \left(f_\delta(y)\right)^p\ \mathrm{d}y\ \le C\ .$$

Therefore, we may extract a subsequence (f_{δ_n}) and find $f \in L^p(Y, y^\sigma\,\mathrm{d}y)$ such that

$$f_{\delta_n} \rightharpoonup f \quad \text{in} \quad L^p(Y, y^\sigma\,\mathrm{d}y) \quad \text{as} \quad \delta_n \to 0\ . \tag{3.18}$$

Clearly,

$$f \ge 0 \qquad \text{and} \qquad M_1(f) = \varrho \tag{3.19}$$

owing to (3.11) and $\sigma \le 1$. In particular, $f \not\equiv 0$ since $\varrho > 0$.

It therefore remains to prove that $L(f) = 0$ a.e. in Y. For that purpose, we observe that (3.13) also reads

$$L(f_\delta) = L(f_\delta) - L_\delta(f_\delta) \quad \text{a.e. in} \quad Y\ . \tag{3.20}$$

On the one hand, we have $L(f_\delta) = \tilde{L}(g_\delta)$, where $g_\delta(y) := y^\sigma\ f_\delta(y)$, $y \in Y$, and $\tilde{L}$ is defined as L with $\tilde{\gamma}(y,y') := y^{-\sigma}\ \gamma(y,y')$ and $\tilde{K}(y,y') := (y\,y')^{-\sigma}\ K(y,y')$ instead of γ and K. Owing to (1.6) and (1.16), L_δ is weakly continuous in $L^1(Y)$ (see, e.g., [13, 15]). We then deduce from this property and the convergence (3.18) that

$$L(f_{\delta_n}) = \tilde{L}(g_{\delta_n}) \rightharpoonup \tilde{L}(g) = L(f) \quad \text{in} \quad L^1(Y)\ , \tag{3.21}$$

with $g(y) := y^\sigma\ f(y)$, $y \in Y$.

On the other hand, let ψ be an arbitrary function in $C_0^\infty(Y)$ and choose $a > 0$ such that the support of ψ is contained in $[a, y_0 - a]$. Then, $\psi(y) \le (\|\psi\|_\infty\ y)/a$. This fact, together with

(1.2), (1.10) and (1.12) allow us to deduce that the functions ψ_{c} and ψ_{s} defined in (2.2) and (2.3), respectively, satisfy

$$
\begin{aligned}
|\psi_{\mathsf{c}}(y,y')| &\le |\psi(y+y')| + |\psi(y)| + |\psi(y')| + \frac{\|\psi\|_\infty\ Q(y,y')}{a} \int_0^{y+y'} y''\ \beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' \\
&\le \frac{3\ \|\psi\|_\infty}{a}\ (y+y')\ , \\
|\psi_{\mathsf{s}}(y,y')| &\le \frac{\|\psi\|_\infty}{a} \int_0^{y_0} y''\ \beta_{\mathsf{s}}(y+y',y'')\ \mathrm{d}y'' + |\psi(y)| + |\psi(y')| \le \frac{2\ \|\psi\|_\infty}{a}\ (y+y')\ .
\end{aligned}
$$

We then infer from (2.2), (2.3), (3.11) and (3.15) that

$$
\begin{aligned}
\left| \int_0^{y_0} \psi(y)\ (L(f_\delta) - L_\delta(f_\delta))(y)\ \mathrm{d}y \right| &\le \frac{\delta}{2} \int_0^{y_0} \int_0^{y_0-y} |\psi_{\mathsf{c}}(y,y')|\ f_\delta(y)\ f_\delta(y')\ \mathrm{d}y'\mathrm{d}y \\
&\quad + \frac{\delta}{2} \int_0^{y_0} \int_{y_0-y}^{y_0} |\psi_{\mathsf{s}}(y,y')|\ f_\delta(y)\ f_\delta(y')\ \mathrm{d}y'\mathrm{d}y \\
&\le \frac{5\ \|\psi\|_\infty\ \delta}{a} \int_0^{y_0} \int_0^{y_0} y\ f_\delta(y)\ f_\delta(y')\ \mathrm{d}y'\mathrm{d}y \\
&\le \frac{5\ \varrho\ \|\psi\|_\infty\ \delta^{1/2}}{a} \left(\delta^{1/2}\ M_0(f_\delta)\right) \\
&\le \frac{C\ \|\psi\|_\infty}{a}\ \delta^{1/2}\ .
\end{aligned}
$$

Letting $\delta \to 0$ then implies that

$$
L(f_\delta) - L_\delta(f_\delta) \to 0 \quad \text{in} \quad \mathcal{D}'(Y) \tag{3.22}
$$

as $\delta \to 0$. As a consequence of (3.21) and (3.22), we may pass to the limit as $\delta_n \to 0$ in (3.20) and conclude that $L(f) = 0$ in $\mathcal{D}'(Y)$. Since $L(f)$ actually belongs to $L^1(Y)$, we conclude that $L(f) = 0$ a.e. in Y, which completes the proof of Theorem 1.1. □

4. Non-existence of non-zero solutions

In this concluding section we show that the problem (1.4), (1.5) is not always well-posed if either (1.13)-(1.15) or (1.11) are violated. To this end, we first assume that there is no spontaneous breakage, i.e. $\gamma \equiv 0$, and that K and P are strictly positive a.e. on their domains. Furthermore, we suppose that shattering and scattering are mass preserving and binary processes, that is, that (1.10) and (1.12) are satisfied and additionally that

$$
\beta_{\mathsf{c}}(y,y') = \beta_{\mathsf{c}}(y,y-y')\ , \qquad 0 < y' < y < y_0\ , \tag{4.1}
$$

$$
\beta_{\mathsf{s}}(y,y') = \beta_{\mathsf{s}}(y,y-y') > 0\ , \quad 0 < y - y_0 < y' < y_0\ , \tag{4.2}
$$

and

$$
\beta_{\mathsf{s}}(y,y') = 0\ , \quad 0 < y' < y - y_0 < y_0\ . \tag{4.3}
$$

The latter assumption is due to consistency of the model since each of the daughter particles y' and $y - y'$ in (4.2) has to belong to Y. Note that (1.10), (1.12) and (4.1)-(4.3) imply

$$
Q(y,y') \int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' = 2\ Q(y,y') \quad \text{for a.a. } (y,y') \in \Xi\ ,
$$

and

$$
\int_{y-y_0}^{y_0} \beta_{\mathsf{s}}(y,y')\ \mathrm{d}y' = 2 \quad \text{for a.a. } y \in (y_0, 2y_0)\ ,
$$

in particular, (1.15) is violated due to the Hölder inequality.

Proposition 4.1. *If $\gamma \equiv 0$, K and P are strictly positive a.e. on their domains and if β_{c} and β_{s} satisfy (4.1)-(4.3), the only solution $f \in L^1_+(Y)$ to (1.4) is $f \equiv 0$.*

Proof. Let $u \in L^1_+(Y)$ be a solution to (1.4). Then we deduce from (2.1)-(2.3) with $\psi \equiv 1$ that

$$\begin{aligned} 0 = & - \int_0^{y_0} L(u)(y)\ \mathrm{d}y \ = \ \frac{1}{2} \int_0^{y_0} \int_0^{y_0-y} K(y,y')\ P(y,y')\ u(y)\ u(y')\ \mathrm{d}y'\mathrm{d}y \\ \geq & \ \frac{1}{2} \int_0^{y_0/2} \int_0^{y_0/2} K(y,y')\ P(y,y')\ u(y)\ u(y')\ \mathrm{d}y'\mathrm{d}y \ \geq \ 0 \end{aligned}$$

whence $u \equiv 0$ on $(0, y_0/2)$ and $L_{\mathsf{c}}(u)(y) = 0$ for a.e. $y \in Y$. Therefore

$$\begin{aligned} 0 = L(u)(y) = & \ \frac{1}{2} \int_{y_0}^{y_0+y} \int_{y'-y_0}^{y_0} K(y'',y'-y'')\ \beta_{\mathsf{s}}(y',y)\ u(y'')\ u(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\ & - \ u(y) \int_{y_0-y}^{y_0} K(y,y')\ u(y')\ \mathrm{d}y' \end{aligned} \tag{4.4}$$

for a.e. $y \in Y$. We claim that this implies that $u \equiv 0$ on $\big(\xi \vee (y_0-\xi), (y_0+\xi)/2\big)$ for a.e. $\xi \in \big(y_0/3, y_0\big)$ such that $u(\xi) = 0$ (recall that $\xi \vee (y_0-\xi) := \max\{\xi, y_0-\xi\}$). Indeed, consider $\xi \in \big(y_0/3, y_0\big)$ such that $u(\xi) = 0$. Since

$$\big(\xi \vee (y_0-\xi), (y_0+\xi)/2\big)^2 \subset \big\{(y'',y')\ ;\ \xi < y'' < y_0\ ,\ y_0 < y'+y'' < y_0+\xi\big\}\ ,$$

we infer from (4.4) that

$$\begin{aligned} 0 = & \int_{y_0}^{y_0+\xi} \int_{y'-y_0}^{y_0} K(y'',y'-y'')\ \beta_{\mathsf{s}}(y',\xi)\ u(y'')\ u(y'-y'')\ \mathrm{d}y''\mathrm{d}y' \\ = & \int_0^{y_0} \int_{y_0}^{y_0+(y''\wedge\xi)} K(y'',y'-y'')\ \beta_{\mathsf{s}}(y',\xi)\ u(y'')\ u(y'-y'')\ \mathrm{d}y'\mathrm{d}y'' \\ = & \int_{\xi}^{y_0} \int_{y_0-y''}^{y_0+\xi-y''} K(y'',y')\ \beta_{\mathsf{s}}(y'+y'',\xi)\ u(y'')\ u(y')\ \mathrm{d}y'\mathrm{d}y'' \\ \geq & \int_{\xi\vee(y_0-\xi)}^{(y_0+\xi)/2} \int_{\xi\vee(y_0-\xi)}^{(y_0+\xi)/2} K(y'',y')\ \beta_{\mathsf{s}}(y'+y'',\xi)\ u(y'')\ u(y')\ \mathrm{d}y'\mathrm{d}y'' \\ \geq & \ 0\ , \end{aligned}$$

whence $u \equiv 0$ on $\big((\xi \vee y_0-\xi), (y_0+\xi)/2\big)$. Defining $\xi_k := (1-2^{-k-1})y_0$, we inductively infer that $u \equiv 0$ on $(0,\xi_k)$ for $k \in \mathbb{N}$ by a density argument, whence $u \equiv 0$ on Y. □

On the other hand, integrating (1.4) over Y and recalling that scattering produces at least two daughter particles, i.e.

$$\int_0^{y_0} \beta_{\mathsf{s}}(y,y')\ \mathrm{d}y' \ \geq \ 2\ , \quad y \in (y_0, 2y_0)\ ,$$

we easily see that zero is the only steady state provided that, in addition,

$$Q(y,y') \int_0^{y+y'} \beta_{\mathsf{c}}(y+y',y'')\ \mathrm{d}y'' \ \geq \ 2\ Q(y,y') + P(y,y')\ , \quad y+y' \in Y\ ,$$

and

$$\int_0^y \left(1-\frac{y'}{y}\right)\ \gamma(y,y')\ \mathrm{d}y' > 0\ , \quad y \in Y\ .$$

Obviously, the former assumption contradicts (1.11).

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