On Diffusive and Non-Diffusive Coalescence and Breakage Processes

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Contents

Abstract

Coalescence and breakage equations describe the evolution of a system consisting of a very large number of particles that can either merge to build larger particles or split into smaller ones. Denoting by Y the set of all possible particle sizes, the continuous coalescence-breakage equations without diffusion take the form

$$\begin{aligned} \partial_t u(t,y) &= f(t,y,u)\ , \quad t>0\ , \quad y \in Y\ , \\ u(0,y) &= u^0(y)\ , \qquad y \in Y\ , \end{aligned} \tag{1}$$

where $u(t,y)$ represents the particle size distribution function. However, if one takes into consideration also diffusion, a diffusion term is added in the above equations and, in addition, the right hand sides may depend on spatial coordinates. This leads to the uncountable set of partial differential equations

$$\begin{aligned} \partial_t u(t,x,y) - d(t,x,y)\Delta_x u(t,x,y) &= f(t,x,y,u)\ , \quad t>0\ , \quad x \in \Omega\ , \quad y \in Y\ , \\ \partial_\nu u(t,x,y) &= 0\ , \qquad t>0\ , \quad x \in \partial\Omega\ , \quad y \in Y\ , \\ u(0,x,y) &= u^0(x,y)\ , \qquad x \in \Omega\ , \quad y \in Y\ , \end{aligned} \tag{2}$$

where $\Omega \subset \mathbb{R}^n$ is a given domain.
In literature, the equations in (1) and (2) are usually formulated for $Y = (0,\infty)$ meaning that particles may become arbitrarily large. The present thesis is devoted to the case when a maximal particle size is presupposed requiring a re-formulation of coalescence of large particles. This particularly for liquid-liquid dispersions realistic assumption being introduced in [**28**] will be developed further in the following. Besides coalescence and breakage also high energy collisions of particles will be considered.

This thesis consists mainly of two independent parts, one of them being dedicated to the (autonomous) ordinary differential equations of (1), the other to the partial differential equations of (2). In both cases existence and uniqueness of positive, mass-preserving solutions is proven. In addition, sufficient conditions for global existence are derived. In the easier situation of the ordinary differential equation (1), also long-time behaviour is studied.

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Zusammenfassung

Koagulations- und Fragmentationsprozesse beschreiben die Evolution eines Systems bestehend aus einer grossen Anzahl von Teilchen, die sich einerseits zu grösseren Teilchen zusammenschliessen oder aber in kleinere zerfallen können. Bezeichnen wir mit Y die Menge aller möglichen Teilchengrössen, so sind die stetigen Koagulations- und Fragmentationsgleichungen unter Vernachlässigung von Diffusion von der Form

$$\begin{aligned} \partial_t u(t,y) &= f(t,y,u)\ , \quad t>0\ , \quad y \in Y\ , \\ u(0,y) &= u^0(y)\ , \qquad y \in Y\ , \end{aligned} \tag{1}$$

wobei $u(t,y)$ die Verteilungsfunktion der Teilchengrösse repräsentiert. Berücksichtigt man hingegen auch Diffusion, dann werden obige Gleichungen durch einen Diffusionsterm ergänzt, und die rechten Seiten können zusätzlich ortsabhängig sein. Dies führt sodann auf überabzählbar viele partielle Differentialgleichungen der Gestalt

$$\begin{aligned} \partial_t u(t,x,y) - d(t,x,y)\Delta_x u(t,x,y) &= f(t,x,y,u)\ , \quad t>0\ , \quad x \in \Omega\ , \quad y \in Y\ , \\ \partial_\nu u(t,x,y) &= 0\ , \quad t>0\ , \quad x \in \partial\Omega\ , \quad y \in Y\ , \\ u(0,x,y) &= u^0(x,y)\ , \quad x \in \Omega\ , \quad y \in Y\ , \end{aligned} \tag{2}$$

wobei $\Omega \subset \mathbb{R}^n$ ein vorgegebenes Gebiet ist.
Betrachtet man die Literatur, so sind die Gleichungen in (1) und (2) meist für $Y = (0,\infty)$ formuliert, so dass Teilchen beliebig gross werden können. Die vorliegende Arbeit hingegen widmet sich dem Fall, wo eine maximale Teilchengrösse als bekannt vorausgesetzt wird, was eine Neuformulierung der Koagulation grosser Teilchen bedingt. Diese insbesondere für Flüssig-Flüssig-Dispersionen realistische und in [**28**] erstmals verwendete Annahme soll im Folgenden weiterentwickelt werden. Neben Koagulation und Fragmentation werden auch hochenergetische Kollisionen von Teilchen betrachtet.

Diese Arbeit besteht im wesentlichen aus zwei voneinander unabhängigen Teilen, wobei der erste davon den (autonomen) gewöhnlichen Differentialgleichungen von (1) gewidmet ist und der zweite jenen partiellen von (2). In beiden Fällen wird die Existenz und Eindeutigkeit von positiven, massenerhaltenden Lösungen bewiesen. Zusätzlich werden hinreichende Bedingungen für globale Existenz angegeben. In der einfacheren Situation der gewöhnlichen Differentialgleichungen (1) wird auch das Langzeitverhalten studiert.

Introduction

Since the pioneering work of von Smoluchowski [**63**], [**64**] dating back to the beginning of the 20th century, the literature on coagulation and fragmentation processes has considerably grown. Originally intended to describe the kinetics of colloids moving according to Brownian motion, that model has since been widely extended. Much effort has been invested in the further development not only of the underlying physical models, but also in their mathematical investigations. In all those models, a system of a very large number of particles is considered which are assumed to be completely identified by their size like mass or volume. This size might be a positive real number in the continuous case or a positive integer in the discrete case. The particles then undergo the influences of coagulation and/or fragmentation, meaning that they can merge to build larger particles or split into smaller ones. Of course, the reasons causing coagulation or fragmentation (or coalescence and breakage in terminology of liquids) depend on the scope of application of these models which arise in a multitude of situations such as astronomy, biology, oil industry, polymer and aerosol science.

In this thesis, attention is focused on an extension of a new model being introduced for the first time by Fasano and Rosso [**28**] (see also [**27**], [**29**]). It describes the evolution of a liquid-liquid dispersion, which is a system formed by two immiscible liquids and where one of these liquids consists of droplets that are finely distributed in the other one. What makes this model particularly interesting is that the experimental observation (for instance, see [**51**]) of a maximal droplet mass (or volume) is taken into account. This maximal droplet size depends on several parameters, but particularly on temperature. In literature, this fact has either been disregarded so far or was introduced only as an artificial cut off (see [**66**]) neglecting a fundamental inconsistency of the model. Indeed, imposing an upper top size for droplet masses requires a new interaction mechanism, which we will call *volume scattering* (or simply *scattering*) in the sequel, in order to prevent the occurrence of droplets resulting from coalescence that are "too large". The underlying idea is rather simple: if two droplets with cumulative mass exceeding the maximal droplet mass collide, the formed cluster is highly unstable and immediately decays in droplets all with mass within the admissible range. As we shall see, this assumption complicates the statement of the problem.
Another new feature taken into consideration in our model is the possibility of high energy collisions leading to a shattering of the involved droplets. Although contemplated in physical literature (cf. [**21**], [**22**], or [**70**]), it has hardly been investigated mathematically so far (however, see [**40**]).

To be more precise, let $u = u(t, y)$ be the distribution function of droplet size at time t (per unit mass), y being the mass (or volume) of a droplet. By $y_0 \in (0, \infty)$ we denote the maximal droplet mass so that $Y := (0, y_0]$ represents in the continuous case the admissible range of droplet masses. Neglecting dependence on spatial coordinates for a moment (which seems to be reasonable in a batch reactor with sufficiently high shear rate, for instance), the evolution of the system of droplets that undergo both coalescence and breakage can be described by the uncountable set of integro-differential equations

$$\begin{aligned} \dot{u}(y) &= \varphi(u)L(u)(y) \ , \quad t > 0 \ , \quad y \in Y \ , \\ u(0, y) &= u^0(y) \ , \qquad\qquad\quad y \in Y \ , \end{aligned} \tag{$*$}$$

where u^0 is a given initial distribution. Here the function

$$L(u) := L_b(u) + L_c(u) + L_s(u)$$

in $(*)$ is defined by

$$\begin{aligned}
L_b(u)(y) &:= \int_y^{y_0} \gamma(y',y)u(y')\ dy' - u(y)\int_0^y \frac{y'}{y}\gamma(y,y')\ dy'\ ,\\
L_c(u)(y) &:= \frac{1}{2}\int_0^y K(y',y-y')P(y',y-y')u(y')u(y-y')\ dy'\\
&\qquad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} K(y'',y'-y'')Q(y'',y'-y'')\beta_c(y',y)u(y'')u(y'-y'')\ dy''dy'\\
&\qquad - u(y)\int_0^{y_0-y} K(y,y')\big\{P(y,y')+Q(y,y')\big\}u(y')\ dy'\ ,\\
L_s(u)(y) &:= \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\beta_s(y',y)u(y'')u(y'-y'')\ dy''dy'\\
&\qquad - u(y)\int_{y_0-y}^{y_0} K(y,y')u(y')\ dy'\ ,
\end{aligned}$$

for $y \in Y = (0, y_0]$.
The operator $L_b(u)$ gives the gain and loss of droplets of mass y due to multiple spontaneous breakage, where the kernel $\gamma(y,y')$ represents the rate at which a droplet of mass y decays into a droplet of mass $y' \in (0,y)$.
When two droplets y and y' with cumulative mass $y+y' \leq y_0$ collide, three different events may arise being described by the collision operator $L_c(u)$. They either coalesce with probability $P(y,y')$, or a shattering of these droplets occurs with probability $Q(y,y')$, or just nothing happens meaning that the droplets remain unchanged. Note that coalescence (and maybe also shattering) seems to be a rare incident since even head-on collisions do not necessarily result in coagulation (see [**56**]). The symmetric function $K(y,y')$ denotes the rate of binary collision and $\beta_c(y+y',y'')$ is the distribution function of products from a particle $y+y'$ shattering after collision. Here β_c depends merely on the cumulative mass $y+y'$ although it would make only a slight difference in the further analysis to allow β_c to depend on each colliding droplet y and y'. The factors $1/2$ come in to compensate for double counting. In accordance with most models considered in literature we take into account only binary collision.
The "scattering" operator $L_s(u)$ represents the interaction of two droplets whose cumulative mass exceeds y_0 and splits immediately into several droplets all with mass in $Y = (0, y_0]$. The distribution function $\beta_s(y+y',y'')$ for $y+y' \in (y_0, 2y_0]$ has an analogue meaning as $\beta_c(y+y',y'')$ for $y+y' \in Y$ above.
Finally, the efficiency factor $\varphi(u)$ linked to some average properties of the dispersion is also a new feature. The idea is to enhance or depress the dynamics while the mechanical structure of the interactions is described by the kernels $\gamma, \beta_c, \beta_s, K, P$, and Q. For instance, $\varphi(u)$ may be of the form

$$\varphi(u) = \Phi\Big(\int_0^{y_0} u(y)\ dy\ ,\int_0^{y_0} y^{2/3}u(y)\ dy\Big)$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}^+$ is a given function. This means that $\varphi(u)$ is related to the total number of droplets and the total surface area. Clearly, no mathematically substantial differences arise if one considers for each process an individual efficiency factor. But to keep the notation simple, we omit this. Instead, a further development would be to introduce a dependence on this quantities of the dispersion in the kernels themselves (as it should be) rather than as an additional factor.

Formally, the continuous coagulation-fragmentation equations without diffusion usually considered in literature can be recovered from $(*)$ by putting $y_0 := \infty, \varphi \equiv 1$, and $P \equiv 1$ (implying that $Q \equiv 0$). In particular, the bilinear operator $L_s(u)$ and the second term of $L_c(u)$ cancel what simplifies the mathematical investigation on the one hand. However, allowing droplets to become arbitrarily large imposes other notable difficulties such as summability. We refer to [**24**] for a survey of the progress in the study of coagulation-fragmentation processes during the first three quarters of the last century and for further literature, but also to [**38**], [**43**]-[**46**], [**48**], [**49**], and [**57**]-[**59**] even though this list is far from being complete.

As mentioned above, the model $(*)$ is adapted from those of [**28**] but includes some extensions. In [**28**] the authors consider the case of pure spontaneous binary breakage only [1], that is, $Q \equiv 0$ in $(*)$ and each droplet decays — if it does — just into two fragments. But if binary breakage is considered, then it is reasonable to assume that

$$\gamma(y,y') = \gamma(y, y-y') \ , \qquad 0 < y' < y \le y_0 \ , \tag{0.1}$$

$$\beta_s(y,y') = \beta_s(y, y-y') \ , \qquad y_0 < y \le 2y_0 \ , \quad y - y_0 < y' \le y_0 \ , \tag{0.2}$$

and

$$\beta_s(y,y') = 0 \ , \quad 0 < y' < y - y_0 \ . \tag{0.3}$$

Indeed, if a droplet of mass y decays into a droplet of mass y', then also a droplet of mass $y - y'$ is formed. On the other hand, each one of the fragments y' and $y - y'$ has to belong to $(0, y_0]$. Therefore, (0.3) is due to consistency of our model. (0.1) implies

$$\int_0^y \frac{y'}{y} \gamma(y,y')\ dy' = \frac{1}{2} \int_0^y \gamma(y,y')\ dy' \ , \quad y \in Y \ ,$$

if both integrals exist. Similarly, presupposed that scattering is a mass-preserving mechanism meaning that

$$\int_0^{y_0} y' \beta_s(y,y')\ dy' = y \ , \quad y_0 < y \le 2y_0 \ ,$$

the equality

$$1 = \int_0^{y_0} \frac{y'}{y} \beta_s(y,y')\ dy' = \frac{1}{2} \int_{y-y_0}^{y_0} \beta_s(y,y')\ dy' \ , \qquad y_0 < y \le 2y_0 \ ,$$

holds according to (0.2) and (0.3). Consequently, under the assumption of pure spontaneous binary breakage which amounts to suppose that (0.1)-(0.3) and $Q \equiv 0$ are valid,

[1] This model has been developed further in [**29**] to include also multiple breakage.

the equations in $(*)$ take the form

$$\begin{aligned}\dot{u}(y) = \varphi(u)\Big\{ &\int_y^{y_0} \gamma(y',y)u(y')\ dy' - \frac{1}{2}u(y)\int_0^y \gamma(y,y')\ dy' \\ &+ \frac{1}{2}\int_0^y K(y',y-y')u(y')u(y-y')\ dy' \\ &- u(y)\int_0^{y_0-y} K(y,y')u(y')\ dy' \\ &+ \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\beta_s(y',y)u(y'')u(y'-y'')\ dy''dy' \\ &- u(y)\int_{y_0-y}^{y_0} K(y,y')u(y')\ dy'\Big\}\end{aligned}$$

for $y \in Y$. Exactly these equations are considered in [**28**]. By using the Arzelà-Ascoli theorem, the authors prove existence and uniqueness of a global non-negative solution which is, in addition, Lipschitz continuous with respect to the droplet size. Of course, such a regularity result requires more regularity from the kernels and from the initial value than actually needed for sole existence. In particular, one has to impose that the kernels and the initial value are (piecewise) continuously differentiable. The results are achieved assuming that the breakage rate

$$y \mapsto \frac{1}{2}\int_0^y \gamma(y,y')\ dy'$$

is a bounded function on $[0,y_0]$. That this is inessential is shown by Borsi [**18**]. Allowing the breakage rate to have a singularity at y_0, he obtains the same results as Fasano and Rosso. Further, numerical simulations for this model are performed by Mancini and Rosso [**42**], who derive some interesting features concerning the qualitative behaviour of solutions. For instance, the asymptotic distribution appears to be independent of the shape of the initial distribution as it is expected from a physical point of view.

The situation now changes drastically if one removes the fundamental assumption of spatial homogeneity and takes into consideration also diffusion. Indeed, even the case when diffusion is described by the Laplace operator, the simplest diffusion operator, becomes a rather tough problem, as we shall see. This may be one of the reasons why only little literature on continuous coagulation-fragmentation processes with diffusion is available. Up to our knowledge, only three articles exist which treat this problem (in the situation where there is no volume scattering, that is, where $y_0 = \infty$). We will return to them subsequently.

If we denote again by $u = u(t,x,y)$ the distribution function of droplet size y at time t and position x, the continuous coalescence-breakage equations taking into account movement due to diffusion read as

$$\begin{aligned}\partial_t u(y) - d(t,x,y)\Delta_x u(y) &= L(t,x,u)(y) && \text{in } \Omega\ , \quad t>0\ , \quad y \in Y\ , \\ \partial_\nu u(y) &= 0 && \text{on } \partial\Omega\ , \quad t>0\ , \quad y\in Y\ , \\ u(0,\cdot,y) &= u^0(y) && \text{in } \Omega\ , \quad y \in Y\ .\end{aligned} \qquad (**)$$

Here Ω is a bounded and smooth domain in $\mathbb{R}^n$, $n \geq 1$, and ν is its outward normal vector. The diffusion coefficient d may depend on t, x, and y although we sometimes restrict this generality for certain results. Moreover, the right hand side of $(**)$ is given by

$$L(t,x,u) := L_b(t,x,u) + L_c(t,x,u) + L_s(t,x,u) \ ,$$

where the operators L_b, L_c, and L_s are defined as above but with kernels $\gamma, \beta_c, \beta_s, K, P$, and Q now depending also on $(t,x) \in \mathbb{R}^+ \times \Omega$. For simplicity, we neglect the efficiency factor $\varphi(u)$ in this setting.

The first paper treating continuous coagulation and fragmentation processes with diffusion has been written by Amann [**10**]. There, the author considers the case when Ω is equal to $\mathbb{R}^n$ so that there are no boundary conditions, but he allows more general diffusion operators than in $(**)$. Interpreting the equations as a Banach-space-valued Cauchy Problem (see below), existence and uniqueness of solutions is proven with the aid of semigroup theory. Moreover, positivity is derived and also global existence is obtained in particular cases.
This idea is taken up by Amann and Weber [**14**] in order to investigate the behaviour of particles being suspended in a carrier fluid. Again, well-posedness (at least local in time) and positivity is shown in the case $\Omega = \mathbb{R}^n$.
A completely different approach choose Laurençot and Mischler [**39**] when $\Omega \subset \mathbb{R}^n$ is bounded. Based on weak and strong compactness methods in L_1, the authors prove global existence (but not uniqueness) of weak solutions in the case of binary fragmentation, additionally assuming either the so-called detailed-balance condition or a monotonicity condition on the coagulation kernel. Furthermore, they study long-time behaviour under the detailed-balance condition.

This thesis is organized as follows. Part 1 is devoted to the study of the ordinary differential equations $(*)$. Besides including multiple breakage and shattering, our contribution consists of proving global existence and uniqueness of positive solutions (cf. chapter 2) under weaker assumptions on the kernels and the initial distribution than made in [**28**]. Of course, we have to accept less regularity. For these results we use a different method than in the cited paper, namely we interpret $(*)$ as an ordinary differential equation in the Banach space $L_1(Y)$. Moreover, long-time behaviour of the obtained solutions is investigated in chapter 3. It is shown (see section 3.1) that sufficient conditions imply that the orbits are relatively weakly compact in $L_1(Y)$. Further, imposing an extended detailed-balance condition and adapting ideas of [**39**], we examine the resulting equilibria for being stable and attractive (cf. section 3.2).
In part 2 the partial differential equations of $(**)$ in a bounded domain $\Omega \subset \mathbb{R}^n$ are considered. A re-formulation allows the treatment of them as an abstract vector-valued Cauchy Problem of the form

$$\dot{u} + A(t)u = L(t,u) \ , \quad t > 0 \ , \qquad u(0) = u^0 \ ,$$

in the space $L_p(\Omega, E)$, where E is a suitable function space over Y. Since recent results of Denk, Hieber, and Prüss [**23**] on maximal regularity of vector-valued elliptic operators entail that $-A(t) := d(t,\cdot,\cdot)\Delta$, subject to Neumann boundary conditions, generates an analytic semigroup on $L_p(\Omega, E)$, most work is invested in the proof of some delicate interpolation results for L_p-spaces involving boundary conditions (see chapter 6). We then derive in chapter 7 existence and uniqueness of positive solutions which, in addition, exist globally in particular situations.

General Notations and Conventions

If not stated otherwise, all vector spaces are over the reals. If there are implicit or explicit references to complex numbers in a given formula, then it is understood that the latter is interpreted as the corresponding complexification.
By c we denote various constants which may differ from occurrence to occurrence, but which are always independent of the free variables. Dependence on additional parameters, say $a, b, \ldots$, we sometimes express by writing $c(a, b, \ldots)$.
If X is a nonempty set and A a subset of X, the symbol χ_A stands for the characteristic function of A (in X), that is, $\chi_A(x) := 1$ if $x \in A$ and $\chi_A(x) := 0$ if $x \in X \setminus A$.
For $J \subset \mathbb{R}$ we put $\dot{J} := J \setminus \{0\}$.
All other notations are usually explained where they appear for the first time.

Part 1

Coalescence and Breakage Processes without Diffusion

1. Preliminaries

In this part we consider the continuous coalescence and breakage processes without diffusion. We therefore assume droplets to be uniformly distributed so that the distribution function $u = u(t, y)$ is independent of spatial coordinates. The evolution of the system of droplets that undergo both coalescence and breakage can then be described by the integro-differential equations

$$\begin{aligned} \dot{u}(y) &= \varphi(u)\big\{L_b[u](y) + L_c[u,u](y) + L_s[u,u](y)\big\} \ , \quad t > 0 \ , \\ u(0,y) &= u^0(y) \end{aligned} \tag{$*$}$$

for $y \in Y := (0, y_0]$, where u^0 is a given initial distribution, and where the operators in $(*)$ are defined as

$$\begin{aligned} L_b[u](y) &:= L_b^1[u](y) - L_b^2[u](y) \\ &:= \int_y^{y_0} \gamma(y', y) u(y') \, dy' - u(y) \int_0^y \frac{y'}{y} \gamma(y, y') \, dy' \ , \\ L_c[u,v](y) &:= L_c^1[u,v](y) + L_c^2[u,v](y) - L_c^3[u,v](y) \\ &:= \frac{1}{2} \int_0^y K(y', y-y') P(y', y-y') u(y') v(y-y') \, dy' \\ &\quad + \frac{1}{2} \int_y^{y_0} \int_0^{y'} K(y'', y'-y'') Q(y'', y'-y'') \beta_c(y', y) u(y'') v(y'-y'') \, dy'' dy' \\ &\quad - u(y) \int_0^{y_0 - y} K(y, y') \big\{P(y, y') + Q(y, y')\big\} v(y') \, dy' \ , \\ L_s[u,v](y) &:= L_s^1[u,v](y) - L_s^2[u,v](y) \\ &:= \frac{1}{2} \int_{y_0}^{2y_0} \int_{y'-y_0}^{y_0} K(y'', y'-y'') \beta_s(y', y) u(y'') v(y'-y'') \, dy'' dy' \\ &\quad - u(y) \int_{y_0 - y}^{y_0} K(y, y') v(y') \, dy' \ . \end{aligned}$$

Often, we simply write $L_h[u]$ instead of $L_h[u,u]$ for $h = c, s$, but keep in mind the definition of $L_h[u,v]$. Moreover, we put

$$L[u] := L_b[u] + L_c[u] + L_s[u] \ .$$

In our approach we interpret $(*)$ as an ordinary differential equation in the Banach space $L_1 := L_1(Y)$. In the sequel we put $|\cdot|_1 := |\cdot|_{L_1}$ and assume throughout this part that the following hypotheses are satisfied:

(H_1) $\varphi : L_1 \to \mathbb{R}^+$ is bounded and Lipschitz continuous on bounded sets;

(H_2) γ is a measurable function from $\Delta := \big\{(y, y') \ ; \ 0 < y' < y \le y_0\big\}$ into $\mathbb{R}^+$ and there exists $m_\gamma > 0$ with

$$\int_0^y \gamma(y, y') \, dy' \le m_\gamma \ , \quad \text{a.a. } y \in Y \ ;$$

(H_3) β_c is a measurable function defined on Δ with values in $\mathbb{R}^+$ and there exists $m_c \geq 2$ with

$$\int_0^{y+y'} \beta_c(y+y',y'')\ dy'' \leq m_c\ , \quad \text{a.a. } (y,y') \in Y \times Y \text{ with } y+y' \in Y\ ,$$

and

$$\int_0^{y+y'} y''\beta_c(y+y',y'')\ dy'' = y+y'\ , \quad \text{a.a. } (y,y') \in Y \times Y \text{ with } y+y' \in Y\ ;$$

(H_4) β_s is a measurable function defined on $\Lambda := \big\{(y,y')\ ;\ 0 < y' \leq y_0 < y \leq 2y_0\big\}$ with values in $\mathbb{R}^+$, and there exists $m_s \geq 2$ with

$$\int_0^{y_0} \beta_s(y+y',y'')\ dy'' \leq m_s\ , \quad \text{a.a. } (y,y') \in Y \times Y \text{ with } y+y' \in (y_0,2y_0]\ ,$$

and

$$\int_0^{y_0} y''\beta_s(y+y',y'')\ dy'' = y+y'\ , \quad \text{a.a. } (y,y') \in Y \times Y \text{ with } y+y' \in (y_0,2y_0]\ ;$$

(H_5) $P,Q,K \in L_\infty(Y \times Y, \mathbb{R}^+)$ are symmetric and $0 \leq P+Q \leq 1$ a.e..

Observe that the first integrals in Hypotheses (H_3) and (H_4) represent the total number of droplets produced by shattering and scattering, respectively, so that they are *a priori* bounded below by 2. Thus, although multiple breakage is allowed in our model, Hypotheses $(H_2)-(H_4)$ imply that only a limited number of daughter droplets are produced by rupture. The second conditions of (H_3) and (H_4) mean conservation of mass. Note that these assumptions are weaker than those of [**28**] (if we put $P \equiv 1$, of course).
With regard to the subsequent study of the long-time behaviour, we restrict ourselves to the autonomous problem, that is, to time-independent kernels.

2. Existence, Uniqueness, and Properties of Solutions

This chapter is devoted to the proof of existence and uniqueness of a maximal non-negative solution which preserves the total mass. In some cases, e.g. in the case of binary breakage or in the case of pure spontaneous breakage, this solution exists globally. Furthermore, some a priori estimates are derived. The results of this chapter are slight modifications of those published in [**69**].

2.1. Existence of Global Solutions

Use the Hypotheses $(H_2) - (H_5)$ and Fubini's theorem to deduce the following lemma which immediately yields local existence.

Lemma 2.1. *The operator $L_b[\cdot] : L_1 \to L_1$ is linear and $L_h[\cdot,\cdot] : L_1 \times L_1 \to L_1$ is bilinear for $h = c, s$. Moreover, for $u, v \in L_1$ the following estimates hold:*

(i) $|L_b[u]|_1 \leq 2m_\gamma |u|_1$,
(ii) $|L_c[u,v]|_1 \leq (m_c + 3)\|K\|_\infty \; |u|_1 \; |v|_1$,
(iii) $|L_s[u,v]|_1 \leq (m_s + 2)\|K\|_\infty \; |u|_1 \; |v|_1$.

Theorem 2.2. *For each $u^0 \in L_1$ there exists a unique maximal solution*

$$u := u(\cdot; u^0) \in C^1(J(u^0), L_1)$$

for $(*)$, where the maximal existence interval $J(u^0)$ is open in $\mathbb{R}^+$.
If $t^+(u^0) := \sup J(u^0) < \infty$ then

$$\lim_{t \nearrow t^+(u^0)} |u(t; u^0)|_1 = \infty \; . \tag{2.1}$$

Moreover, the map $\big[(t, u^0) \mapsto u(t; u^0)\big]$ generates a semiflow on L_1.

Proof. Since $L_b[\cdot]$ is linear and $L_h[\cdot,\cdot]$ is bilinear for $h \in \{c, s\}$, Hypothesis (H_1) and Lemma 2.1 imply that the map

$$L_1 \to L_1 \; , \quad u \mapsto \varphi(u)\big\{L_b[u] + L_c[u,u] + L_s[u,u]\big\}$$

is Lipschitz continuous and bounded on bounded sets. Thus, standard arguments from the theory of ordinary differential equations (see [**7**]) lead to the assertion. □

The solution is actually more regular with respect to time.

Corollary 2.3. *For any $u^0 \in L_1$ it holds $u = u(\cdot; u^0) \in C^{2-}(J(u^0), L_1)$. Moreover, if $\varphi \equiv const$ then $u \in C^\infty(J(u^0), L_1)$.*

Proof. The first assertion is a consequence of the fact that u as well as the right hand side of $(*)$ are Lipschitz continuous and hence $\dot{u} \in C^{1-}(J(u^0), L_1)$.
Assume now that $\varphi \equiv const$. Since $u \in C^1(J(u^0), L_1)$ and $L_b[\cdot]$ is linear and continuous, $L_b[u]$ belongs to $C^1(J(u^0), L_1)$ with

$$\frac{d}{dt} L_b[u(t)] = L_b[\dot{u}(t)] \; , \quad t \in J(u^0) \; ,$$

and similarly $L_h[u,u] \in C^1(J(u^0), L_1)$ with

$$\frac{d}{dt} L_h[u(t), u(t)] = L_h[\dot{u}(t), u(t)] + L_h[u(t), \dot{u}(t)] \; , \quad t \in J(u^0) \; , \quad h = c, s \; .$$

The right hand side of $(*)$ is therefore continuously differentiable and we conclude that $u \in C^2(J(u^0), L_1)$. The assertion follows now by induction. □

In the sequel, for $u^0 \in L_1$ given, we denote by $u = u(\cdot; u^0) \in C^1(J(u^0), L_1)$ the unique maximal solution for $(*)$, with the understanding that

$$u(t,y) := u(t;u^0)(y) \ , \quad t \in J(u^0) \ , \quad y \in Y \ .$$

If no confusion seems likely we sometimes suppress any of the variables t and y in a given formula. We also put

$$\varphi(t) := \varphi\big(u(t)\big) \ , \quad t \in J(u^0) \ .$$

Furthermore, L_1^+ is the closed subset of L_1 consisting of all $v \in L_1$ which are non-negative almost everywhere.

Theorem 2.4. *For any initial distribution $u^0 \in L_1^+$, the solution $u(t;u^0)$ remains non-negative, i.e., $u(t;u^0) \in L_1^+$ for $t \in J(u^0)$.*

Proof. We choose any $T_0 \in \dot{J}(u^0)$ and put

$$\|\varphi\|_\infty := \max_{0 \le t \le T_0} |\varphi(t)| \ , \quad \|u\|_\infty := \max_{0 \le t \le T_0} |u(t)|_1 \ ,$$

as well as

$$\omega := \|\varphi\|_\infty \big(m_\gamma + \|K\|_\infty \ \|u\|_\infty\big) \ge 0 \ .$$

For $0 \le t \le T \le T_0$ and $v \in L_1$ set

$$\begin{aligned} G(t,v) := \varphi(t)\big\{ L_b[v] + L_c^1[v,v] + L_c^2[v,v] + L_s^1[v,v] \\ - L_c^3[v,u(t)] - L_s^2[v,u(t)]\big\} + \omega v \ . \end{aligned} \tag{2.2}$$

Then $G\big(\cdot, v(\cdot)\big) \in C([0,T], L_1)$ provided $v \in C([0,T], L_1)$ due to Lemma 2.1. Further, there exists $c(T_0) > 0$ with

$$|G(t,v) - G(t,w)|_1 \le c(T_0)\big(1 + |v|_1 + |w|_1\big)|v - w|_1 \tag{2.3}$$

for $v, w \in L_1$ and $0 \le t \le T_0$. Since $v \in L_1^+$ implies

$$L_b^1[v] \ , L_c^2[v,v] \ , L_h^1[v,v] \ \in L_1^+ \ , \quad h = c, s \ ,$$

it follows that

$$G(t,v) \ge -\|\varphi\|_\infty\big(m_\gamma + \|K\|_\infty \|u\|_\infty\big)v + \omega v = 0 \quad \text{a.e.} \tag{2.4}$$

for any $v \in L_1^+$. Now put $p := \|u\|_\infty + 2$ and choose $T \in (0, T_0]$ such that $c(T_0)(p+p^2)T < 1$. Due to (2.3), the definition of

$$F(v)(t) := e^{-\omega t}u^0 + \int_0^t e^{-\omega(t-s)} G\big(s, v(s)\big)\, ds \ , \quad 0 \le t \le T \ ,$$

for

$$v \in \mathcal{V}_T := \big\{ v \in C([0,T], L_1) \ ; \ |v(t)|_1 \le p \ , \ 0 \le t \le T \big\}$$

yields a contraction $F : \mathcal{V}_T \to \mathcal{V}_T$ with constant $c(T_0)(1+2p)T < c(T_0)(p+p^2)T < 1$. On the other hand, u, being a solution of $(*)$, solves

$$\begin{aligned} \dot{v} + \omega v &= G(t,v) \ , \quad 0 < t \le T \ , \\ v(0) &= u^0 \ , \end{aligned}$$

as well and belongs to $\mathcal{V}_T$. Thus, u is the unique fixed point of F. Putting

$$u_0 := u^0 \in \mathcal{V}_T \ , \qquad u_{n+1} := F(u_n) \in \mathcal{V}_T \ , \quad n \in \mathbb{N} \ ,$$

we have by induction and (2.4) that $u_n(t) \geq 0$ a.e. for all $n \in \mathbb{N}$ and $0 \leq t \leq T$. Since $u_n \to u$ in $\mathcal{V}_T$, this implies $u(t) \geq 0$ a.e. for $0 \leq t \leq T$.
Put $T^* := \sup\{\tau \in (0, T_0]\ ;\ u(t) \geq 0 \text{ a.e. for } 0 \leq t \leq \tau\}$, assume $T^* < T_0$, and consider

$$\begin{aligned} \dot{v} + \omega v &= G(t + T^*, v)\ , \quad 0 < t \leq T_0 - T^*\ , \\ v(0) &= u(T^*)\ . \end{aligned} \tag{2.5}$$

Then $u(T^*) \in L_1^+$ since L_1^+ is closed in L_1 and, further, $u(\cdot + T^*)$ is a solution of (2.5). By repeating the above arguments we conclude $u(t + T^*) \geq 0$ a.e., $0 \leq t \leq \tau$, for a suitable $\tau > 0$. But this contradicts our choice of T^*. Therefore, $T^* = T_0$ and, $T_0 \in \dot{J}(u^0)$ being arbitrary, the assertion follows. ◻

Corollary 2.5. *Let $u^0 \in L_1$.*

(i) *If $u^0 > 0$ a.e. then $u(t; u^0) > 0$ a.e. for $t \in J(u^0)$.*
(ii) *If $u^0 \geq r_0$ a.e. for some $r_0 \in (0, \infty)$, then there exists for each $T_0 \in \dot{J}(u^0)$ some $R := R(T_0) > 0$ such that $u(t; u^0) \geq R$ a.e. for $0 \leq t \leq T_0$.*

Proof. Fix $T_0 \in \dot{J}(u^0)$ arbitrarily, choose $\omega := \omega(T_0) \geq 0$ and $T \in (0, T_0]$ as in the proof of Theorem 2.4 and define G by (2.2). Since $u(s) \in L_1^+$ for all $s \in J(u^0)$ and hence $G\big(s', u(s)\big) \in L_1^+$, $s', s \in J(u^0)$, we conclude

$$u(t) = e^{-\omega t} u^0 + \int_0^t e^{-\omega(t-s)} G\big(s, u(s)\big)\ ds \geq e^{-\omega T} u^0 \quad \text{a.e.}\ , \quad 0 \leq t \leq T\ .$$

We may repeat this argument with $u(T) \geq e^{-\omega T} u^0$ a.e. and $G(\cdot + T, \cdot)$ instead of u^0 and G, respectively, to deduce

$$u(t) \geq e^{-2\omega T} u^0 \quad \text{a.e.}\ , \quad T \leq t \leq \min\{2T, T_0\}\ .$$

Inductively, this proves the assertion. ◻

Remark 2.6. Theorem 2.2 and Theorem 2.4 guarantee that the map

$$\big[(t, u^0) \mapsto u(t; u^0)\big]$$

generates a semiflow on L_1^+.

Lemma 2.7. *For any $f \in L_\infty(Y)$ and $v \in L_1$ the following identities hold:*

(i)

$$\int_0^{y_0} f(y) L_b[v](y)\ dy = \int_0^{y_0} \int_0^y \Big\{f(y') - \frac{y'}{y} f(y)\Big\} \gamma(y, y')\ dy'\ v(y)\ dy\ ,$$

(ii)

$$\begin{aligned} &\int_0^{y_0} f(y) L_c[v](y)\ dy \\ &\quad = \frac{1}{2} \int_0^{y_0} \int_0^{y_0 - y} \Big\{P(y, y') f(y + y') - \big[f(y) + f(y')\big]\big[P(y, y') + Q(y, y')\big] \\ &\qquad + Q(y, y') \int_0^{y + y'} f(y'') \beta_c(y + y', y'')\ dy''\Big\} K(y, y') v(y') v(y)\ dy' dy\ , \end{aligned}$$

(*iii*)

$$\int_0^{y_0} f(y)L_s[v](y)\,dy$$
$$= \frac{1}{2}\int_0^{y_0}\int_{y_0-y}^{y_0}\Big\{\int_0^{y_0} f(y'')\beta_s(y+y',y'')\,dy'' - f(y) - f(y')\Big\}K(y,y')v(y')v(y)\,dy'dy\ .$$

Proof. The statements are consequences of Fubini's theorem and suitable changes of variables whereby all of the integrals remain finite due to Hypotheses $(H_2)-(H_5)$. For (*ii*) and (*iii*) recall that K, P, and Q are symmetric. □

Remark 2.8. Suppose $f \equiv 1$. Then, for $v \in L_1^+$, Lemma 2.7 reflects the intuitively evident facts that breakage and scattering increase the total number of droplets. If only binary breakage is considered then scattering does not alter the number of droplets since (H_3) and (0.2) imply

$$\int_0^{y_0} \beta_s(y+y',y'')\,dy' = 2\ ,\quad \text{a.a. } (y,y')\in Y\times Y \text{ with } y+y'\in(y_0,2y_0]\ .$$

Therefore, in this case

$$\int_0^{y_0} L_s[v](y)\,dy = 0\ .$$

Whether collision increases or decreases the total number of clusters depends on the probability of coalescence and on the number of fragments resulting from shattering. However, if $P\equiv 1$ or if two colliding droplets break into two fragments only, the total number of droplets is reduced by this mechanism.

Lemma 2.7 implies that any solution of $(*)$ conserves the total mass.

Theorem 2.9. *Let $u^0\in L_1$. Then, for any $t\in J(u^0)$,*

$$\int_0^{y_0} yu(t;u^0)(y)\,dy = \int_0^{y_0} yu^0(y)\,dy\ .$$

Proof. For $t\in J(u^0)$ we have

$$u(t) = u^0 + \int_0^t \varphi(\sigma)\big\{L_b[u(\sigma)] + L_c[u(\sigma)] + L_s[u(\sigma)]\big\}\,d\sigma\ .$$

Thus [**34**, p.69 f] gives

$$u(t,y) = u^0(y) + \int_0^t \varphi(\sigma)\big\{L_b[u(\sigma)](y) + L_c[u(\sigma)](y) + L_s[u(\sigma)](y)\big\}\,d\sigma \tag{2.6}$$

for a.a. $y\in Y$. Multiplying both sides with y, integrating then over Y, and changing the order of integration, Lemma 2.7 leads to the assertion in view of (H_3) and (H_4). □

Theorem 2.10. *Assume $\|\varphi\|_\infty := \sup_{v\in L_1^+}\varphi(v) < \infty$. Furthermore, let one of the following conditions be satisfied:*

(i) *$K(y,y')\le K^*(y+y')$ for a.a. $(y,y')\in Y\times Y$ and some $K^*>0$;*

(ii) *there exists $z_0\in Y$ such that for a.a. $(y,y')\in Y\times Y$ with $y+y'\le z_0$*

$$\int_0^{y+y'} \beta_c(y+y',y'')\,dy'' \le 2 + \frac{P(y,y')}{Q(y,y')}\ . \tag{2.7}$$

Then the solution $u(\cdot;u^0)$ exists globally for $u^0\in L_1^+$, that is, $J(u^0)=\mathbb{R}^+$.

PROOF. In analogy to the proof of Theorem 2.9 we have

$$\begin{aligned} |u(t)|_1 &= \int_0^{y_0} u(t,y)\ dy \\ &= |u^0|_1 + \int_0^t \varphi(\sigma) \int_0^{y_0} \big\{ L_b[u(\sigma)](y) + L_c[u(\sigma)](y) + L_s[u(\sigma)](y) \big\}\ dyd\sigma \end{aligned} \tag{2.8}$$

for $t \in J(u^0)$ since $u(t)$ is non-negative. Lemma 2.7 leads to the estimate

$$\int_0^t \varphi(\sigma) \int_0^{y_0} L_b[u(\sigma)](y)\ dyd\sigma \le \|\varphi\|_\infty m_\gamma \int_0^t |u(\sigma)|_1\ d\sigma\ .$$

Using

$$K(y,y') \le \frac{\|K\|_\infty}{y_0}(y+y')\ , \quad \text{a.a. } (y,y') \text{ with } y+y' > y_0 \tag{2.9}$$

and conservation of mass we see that

$$\int_0^t \varphi(\sigma) \int_0^{y_0} L_s[u(\sigma)](y)\ dyd\sigma \le \|\varphi\|_\infty m_s \frac{\|K\|_\infty}{y_0} \int_0^{y_0} yu^0(y)\ dy\ \int_0^t |u(\sigma)|_1\ d\sigma\ .$$

If (i) holds then Lemma 2.7 ensures that

$$\int_0^t \varphi(\sigma) \int_0^{y_0} L_c[u(\sigma)](y)\ dyd\sigma \le \|\varphi\|_\infty (m_c+2) K^* \int_0^{y_0} yu^0(y)\ dy\ \int_0^t |u(\sigma)|_1\ d\sigma\ ,$$

since $0 \le P+Q \le 1$. On the other hand, if (ii) is satisfied then

$$\int_A \big\{ -P(y,y') + Q(y,y')\big(\nu(y,y')-2\big)\big\} K(y,y') u(y') u(y)\ d(y,y') \le 0$$

where we put $A := \big\{ (y,y') \in Y \times Y\ ;\ y+y' \le z_0 \big\}$ and

$$\nu(y,y') := \int_0^{y+y'} \beta_c(y+y',y'')\ dy''\ .$$

For $B := \big\{ (y,y') \in Y \times Y\ ;\ z_0 < y+y' \le y_0 \big\}$ we have $K(y,y') \le \|K\|_\infty (y+y')/z_0$ for a.a. $(y,y') \in B$, and as a consequence

$$\begin{aligned} &\int_0^t \varphi(\sigma) \int_0^{y_0} L_c[u(\sigma)](y)\ dyd\sigma \\ &\qquad \le \int_0^t \varphi(\sigma) \int_B \big| -P(y,y') + Q(y,y')\big(\nu(y,y')-2\big)\big| K(y,y') u(y') u(y)\ d(y,y') d\sigma \\ &\qquad \le \|\varphi\|_\infty (m_c+2) \frac{\|K\|_\infty}{z_0} \int_0^{y_0} yu^0(y)\ dy\ \int_0^t |u(\sigma)|_1\ d\sigma\ . \end{aligned}$$

From (2.8) we thus conclude in both cases that

$$|u(t)|_1 \le |u^0|_1 + c_0 \int_0^t |u(\sigma)|_1\ d\sigma\ , \quad t \in J(u^0)\ ,$$

where $c_0 := c_0(|u^0|_1) > 0$ does not depend on $t \in J(u^0)$. Therefore, Gronwall's inequality and Theorem 2.2 lead to the assertion. $\square$

Remarks 2.11. (a) Note that if only spontaneous breakage is allowed meaning that $Q \equiv 0$, then the solution $u(\cdot; u^0)$ for $u^0 \in L_1^+$ is global since (2.7) holds. If binary breakage is considered, in particular, if $\beta_c(y,y') = \beta_c(y, y-y')$, $0 < y' < y \leq y_0$, then Hypothesis (H_3) implies

$$\int_0^{y+y'} \beta_c(y+y', y'')\, dy'' = 2 \ , \quad \text{a.a. } y+y' \in Y \ , \tag{2.10}$$

and whence (2.7) so that we have global existence in this case as well.
(b) Obviously, Problem $(*)$ does not always possess a global solution. For instance, if there exists a constant $\varepsilon_0 > 0$ such that

$$\int_0^{y+y'} \beta_c(y+y', y'')\, dy'' \geq 2 + \frac{P(y,y') + \varepsilon_0}{Q(y,y')} \ , \quad \text{a.a. } y+y' \in Y \ ,$$

and

$$\int_0^{y_0} \beta_s(y+y', y'')\, dy'' \geq 2 + \varepsilon_0 \ , \quad \text{a.a. } y+y' \in (y_0, 2y_0] \ ,$$

then for collision kernels satisfying $K(y,y') \geq K_* > 0$ for a.a. $(y,y') \in Y \times Y$, one has

$$\frac{d}{dt}|u(t;u^0)|_1 \geq \frac{\varepsilon_0}{2} K_* |u(t;u^0)|_1^2 \ , \quad t \in J(u^0) \ ,$$

and hence $\sup J(u^0) < \infty$ if $u^0 \in L_1^+ \setminus \{0\}$.

2.2. A Priori Estimates

This section is devoted to upper and lower a priori estimates for the L_1-norm of the solution which lead to stability or instability of the trivial solution for certain kernels. Bearing in mind that this norm corresponds to the total number of droplets, it is not surprising that its evolution is strongly related to coalescence and breakage of small droplets. Indeed, as we shall see, if small droplets have a rather high coalescence efficiency as well as being stable concerning spontaneous breakage, an upper a priori bound of this norm is valid. On the other hand, a high breakage rate of small droplets implies a lower a priori bound of this norm.

To shorten notation, the moments

$$M_\alpha(t) := \int_0^{y_0} y^\alpha u(t,y)\, dy \ , \quad t \in J(u^0) \ ,$$

are introduced for $\alpha \geq 0$. Hence $M_0(t) = |u(t)|_1$ represents the total number of droplets at time $t \in J(u^0)$ and $M_1(t) \equiv M_1(0)$ is equal to the total mass.
Note that the trivial solution is not attractive for the semiflow $\big[(t,u^0) \mapsto u(t;u^0)\big]$ generated on L_1^+ since conservation of mass yields

$$M_0(t) \geq \frac{M_1(0)}{y_0} \ , \quad t \in J(u^0) \ .$$

First we assume that there exists some small $y_c \in (0, y_0)$ such that droplets with mass less than y_c are not produced by rupture, i.e.

$$\gamma(y,y') = \beta_c(y,y') = \beta_s(\tilde{y}, y') = 0 \ , \quad 0 < y' \leq y_c \ . \tag{2.11}$$

Observe that this also implies that small droplets do not break. Indeed, considering for instance only binary breakage then (2.11) implies

$$\gamma(y,y') = \beta_c(y,y') = 0\ , \quad y < 2y_c\ ,$$

since otherwise at least one of the daughter droplets would have a mass less than y_c. Consequently, droplets with mass less than y_c are those already existing at time $t = 0$ and can disappear only due to coalescence. This fact implies an upper bound for the total number of droplets and stability of the trivial solution as shown in the next proposition.

Proposition 2.12. *Assume that (2.11) holds. Then $u(\cdot;u^0)$ exists globally and*

$$|u(t;u^0)|_1 \le \Big(1+\frac{y_0}{y_c}\Big)|u^0|_1\ , \quad t \ge 0\ .$$

Proof. Condition (2.11) yields

$$\int_0^{y_c} L_b[u](y)\ dy = 0$$

and

$$\int_0^{y_c} L_s[u](y)\ dy = -\int_0^{y_c}\int_{y_0-y}^{y_0} K(y,y')u(y')u(y)\ dy'dy \le 0\ .$$

Moreover, since (2.11) also ensures

$$\begin{aligned}\int_0^{y_c} L_c[u](y)\ dy \le \frac{1}{2}\int_0^{y_c}\int_0^{y_c-y} & P(y,y')K(y,y')u(y')u(y)\ dy'dy \\ & -\frac{1}{2}\int_0^{y_c}\int_0^{y_0-y}\big[P(y,y')+Q(y,y')\big]K(y,y')u(y')u(y)\ dy'dy \\ \le 0\ ,\end{aligned}$$

we conclude from (2.6) that

$$\int_0^{y_c} u(t,y)\ dy \le \int_0^{y_c} u^0(y)\ dy\ , \quad t \in J(u^0)\ .$$

But this entails

$$\begin{aligned}|u(t)|_1 &= \int_0^{y_c} u(t,y)\ dy + \int_{y_c}^{y_0} u(t,y)\ dy \\ &\le \int_0^{y_c} u^0(y)\ dy + \frac{1}{y_c}\int_0^{y_0} yu(t,y)\ dy \le \Big(1+\frac{y_0}{y_c}\Big)|u^0|_1\end{aligned}$$

for all $t \in J(u^0)$, and consequently $J(u^0) = \mathbb{R}^+$ by Theorem 2.2. □

Lemma 2.13. *Let $a, b \ge 0$ with $(a,b) \ne (0,0)$, $c > 0$ and $f^0 > 0$ be given. Put $D := b^2 + 4ac > 0$ and $R := (b+\sqrt{D})/2c$. Then, the unique solution of*

$$\dot{f} = a + bf - cf^2\ , \quad t > 0\ , \qquad f(0) = f^0$$

is given by

$$f(t) = \begin{cases} \frac{b}{2c} + \frac{\sqrt{D}}{2c}\coth\Big(\frac{\sqrt{D}}{2}t + \operatorname{arcoth}\big(\frac{2cf^0-b}{\sqrt{D}}\big)\Big) & \text{if } f^0 > R\ , \\ \frac{b+\sqrt{D}}{2c} & \text{if } f^0 = R\ , \\ \frac{b}{2c} + \frac{\sqrt{D}}{2c}\tanh\Big(\frac{\sqrt{D}}{2}t + \operatorname{artanh}\big(\frac{2cf^0-b}{\sqrt{D}}\big)\Big) & \text{if } 0 < f^0 < R\ , \end{cases}$$

for all $t \ge 0$.

PROOF. Note that f is well-defined. Thus the assertion follows by verification. □

Based on the preceding lemma we are able to establish several estimates for the total number of droplets.

Theorem 2.14. *Suppose that $0 < \varphi_* \le \varphi(v) \le \varphi^* < \infty$ for $v \in L_1^+$ and assume that*

(i) $0 < K_* \le K(y,y')$ *for a.a.* $(y,y') \in Y \times Y$;

(ii) *there exist* $\bar\gamma > 0$ *and* $\sigma \ge 0$ *such that*

$$\int_0^y \Big(1 - \frac{y'}{y}\Big)\gamma(y,y')\ dy' \le \bar\gamma y^\sigma\ , \quad a.a.\ y \in Y\ ; \tag{2.12}$$

(iii) *there exist* $z_0 \in Y$ *and* $\varepsilon > 0$ *such that*

$$\int_0^{y+y'} \beta_c(y+y',y'')\ dy'' \le 2 + \frac{P(y,y') - \varepsilon}{Q(y,y')} \tag{2.13}$$

for a.a. $(y,y') \in Y \times Y$ *with* $y + y' \le z_0$.

Then there exist $c > 0$, depending on φ and the kernels only, and $\mu := \mu(\sigma) \ge 0$ such that

$$|u(t;u^0)|_1 \le c\big(|u^0|_1 + |u^0|_1^\mu\big)\ , \quad t \ge 0\ ,$$

where $\mu > 0$ if $\sigma > 0$.

PROOF. First observe that $J(u^0) = \mathbb{R}^+$ by Theorem 2.10 and (2.13). Next we integrate $(*)$ with respect to y. From Lemma 2.7 and (2.12) we deduce

$$\int_0^{y_0} L_b[u](y)\ dy \le \bar\gamma M_\sigma(t)\ , \quad t \ge 0\ .$$

If $\sigma > 0$ then we choose $\alpha \in \big(0, \min\{1,\sigma\}\big)$, otherwise we put $\alpha := 0$. Then Hölder's inequality yields for $\beta := (\sigma - \alpha)/(1-\alpha)$ and $t \ge 0$

$$M_\sigma(t) \le M_1(0)^\alpha M_\beta(t)^{1-\alpha} \le y_0^{\sigma-\alpha} M_1(0)^\alpha M_0(t)^{1-\alpha}\ .$$

By defining the sets A and B and the function ν as in the proof of Theorem 2.10 we obtain with the aid of Lemma 2.7, (2.13), and conservation of mass

$$\begin{aligned}
&\int_0^{y_0} L_c[u](y)\ dy \\
&\qquad \le \frac12 \int_A \big\{ - P(y,y') + Q(y,y')\big(\nu(y,y') - 2\big)\big\} K(y,y')u(y')u(y)\ d(y,y') \\
&\qquad\quad + \frac12 \int_B \big| - P(y,y') + Q(y,y')\big(\nu(y,y') - 2\big)\big| K(y,y')u(y')u(y)\ d(y,y') \\
&\qquad \le -\frac{\varepsilon}{2} \int_A K(y,y')u(y')u(y)\ d(y,y') + (m_c + 2)\frac{\|K\|_\infty}{z_0} M_1(0)M_0(t)
\end{aligned} \tag{2.14}$$

for $t \ge 0$. Moreover, since (i) holds, it follows for $C := Y^2 \setminus A$ that

$$\begin{aligned}
&-\frac{\varepsilon}{2}\int_A K(y,y')u(y')u(y)\ d(y,y') \\
&\qquad = -\frac{\varepsilon}{2}\int_{Y^2} K(y,y')u(y')u(y)\ d(y,y') + \frac{\varepsilon}{2}\int_C K(y,y')u(y')u(y)\ d(y,y') \\
&\qquad \le -\frac{\varepsilon}{2} K_* M_0(t)^2 + \varepsilon \frac{\|K\|_\infty}{z_0} M_1(0) M_0(t)
\end{aligned}$$

for $t \geq 0$. This and (2.14) yield

$$\int_0^{y_0} L_c[u](y)\, dy \leq -a_1 M_0(t)^2 + a_2 M_1(0) M_0(t) \ , \quad t \geq 0 \ ,$$

with $a_i > 0$ being independent of $t \geq 0$ and $u^0 \in L_1^+$. Similarly, the estimate

$$\int_0^{y_0} L_s[u](y)\, dy \leq (m_s - 2) \frac{\|K\|_\infty}{y_0} M_1(0) M_0(t) \ , \quad t \geq 0 \ ,$$

can be achieved. Putting these facts together and recalling that φ is bounded from below and above, we conclude that M_0 obeys

$$\dot{M}_0(t) \leq b_1 M_1(0)^\alpha M_0(t)^{1-\alpha} - b_2 M_0(t)^2 + b_3 M_1(0) M_0(t) \ , \quad t \geq 0 \ , \tag{2.15}$$

where the constants $b_i > 0$ depend neither on $t \geq 0$ nor on $u^0 \in L_1^+$. Next observe that the function $h(z) := az^{1-\alpha} - bz^2$, $z \geq 0$, with $a, b > 0$ and $0 \leq \alpha < 1$, satisfies

$$h(z) \leq c(\alpha, b) a^{\frac{2}{1+\alpha}} \ , \quad z \geq 0 \ ,$$

for some $c(\alpha, b) > 0$. Therefore, we can estimate the right hand side of (2.15) to obtain

$$\dot{M}_0(t) \leq b_4 M_1(0)^{\frac{2\alpha}{1+\alpha}} - \frac{b_2}{2} M_0(t)^2 + b_3 M_1(0) M_0(t) \ , \quad t \geq 0 \ , \tag{2.16}$$

with $b_4 > 0$. Since $\coth|_{\mathbb{R}^+}$ is decreasing and tanh is bounded by 1, Lemma 2.13 applied to (2.16) implies either $M_0(t) \leq M_0(0)$, $t \geq 0$, provided $M_0(0)$ is sufficiently large, or, otherwise,

$$M_0(t) \leq c_1 M_1(0) + c_2 \sqrt{M_1(0)^2 + M_1(0)^{\frac{2\alpha}{1+\alpha}}} \leq c_3 \big(M_1(0) + M_1(0)^{\frac{\alpha}{1+\alpha}} \big) \ , \quad t \geq 0 \ ,$$

with constants $c_i > 0$ depending only on the b_j's. Since $M_1(0) \leq y_0 M_0(0)$ the assertion follows by setting $\mu(\sigma) := \alpha/(1+\alpha)$. □

Remarks 2.15. (a) If $\sigma = 0$ then assumption (ii) of Theorem 2.14 is redundant in view of Hypothesis (H_2). However, Theorem 2.14 gives a uniform bound for the total number of droplets while, in the case where $\sigma > 0$, it even leads to stability of the trivial solution for the semiflow generated on L_1^+. Likewise, if no spontaneous breakage occurs, i.e. $\gamma \equiv 0$, then one can choose $\sigma > 0$ arbitrarily, of course. In this case, if, in addition, only binary breakage is considered, it is easily seen that

$$\begin{aligned} \dot{M}_0(t) &= \varphi(t) \int_0^{y_0} L_c[u](y)\, dy \\ &= -\frac{1}{2} \varphi(t) \int_0^{y_0} \int_0^{y_0 - y} P(y, y') K(y, y') u(y') u(y)\, dy' dy \leq 0 \end{aligned}$$

for $t \geq 0$, that is, the total number of droplets decreases with time, and it remains constant if also $P \equiv 0$.
(b) In liquid-liquid dispersions, conditions like (2.12) seem to be quite natural if droplets are assumed to be spherical. For further explanation and special kernels satisfying the hypotheses of Theorem 2.14 we refer to Examples 2.21.
(c) Condition (2.13) is fulfilled if either $P \equiv 1$ or $P(y, y') \geq \varepsilon > 0$ for a.a. $y + y' \leq z_0$ and only binary breakage occurs (see (2.10)).

A consequence of the two preceding theorems is the following corollary.

Corollary 2.16. *Let the assumptions of Theorem 2.12 or of Theorem 2.14 be satisfied. Then* $u(\cdot;u^0)\in BC^1(\mathbb{R}^+,L_1)$.

PROOF. Since in both cases $|u(t)|_1\le c(u^0),\ t\ge 0$, for some $c(u^0)>0$, and therefore $\sup_{t\ge 0}\varphi(t)<\infty$ according to Hypothesis (H_1), we obtain from $(*)$ and Lemma 2.1

$$|\dot u(t)|_1\le c\sup_{t\ge 0}\varphi(t)\big(c(u^0)+c(u^0)^2\big)<\infty\ ,\quad t\ge 0\ .$$

□

Remark 2.17. Any bounded solution u for $(*)$ in $C^k(\mathbb{R}^+,L_1)$ for some $k\ge 1$ belongs automatically to $BC^k(\mathbb{R}^+,L_1)$ provided φ is bounded if $k=1$ and $\varphi\equiv const$ if $k>1$. This can be shown inductively. For instance, compare this with Corollary 2.3.

In contrast to the preceding considerations, we now assume the breakage action to be rather effective for small droplets, in the sense that sufficiently many droplets are produced by rupture. In fact, we suppose that the spontaneous breakage frequency is bounded below, which means that also small droplets decay spontaneously at a minimal rate. Consequently, the following two theorems do not apply to the case of pure collisional breakage, i.e. $\gamma\equiv 0$.

Theorem 2.18. *Suppose that* $0<\varphi_*\le\varphi(v)\le\varphi^*<\infty$ *for* $v\in L_1^+$ *and that*

$$\int_0^y\Big(1-\frac{y'}{y}\Big)\gamma(y,y')\ dy'\ge\gamma_*>0\ ,\quad a.a.\ y\in Y\ . \tag{2.17}$$

Then there exists $c_0>0$*, depending only on* φ *and the kernels, such that*

$$\liminf_{t\nearrow t^+(u^0)}|u(t;u^0)|_1\ge c_0\ ,\quad u^0\in L_1^+\setminus\{0\}\ .$$

PROOF. Due to Theorem 2.2 we may assume $t^+(u^0)=\infty$. Integrating $(*)$ with respect to y and applying Lemma 2.7 we obtain the differential inequality

$$\dot M_0(t)\ge\varphi_*\gamma_*M_0(t)-\frac{m_c+2}{2}\varphi^*\|K\|_\infty M_0(t)^2\ ,\quad t\ge 0\ ,$$

where we additionally used the positivity of u and

$$\int_0^{y_0}L_s[u,u](y)\ dy\ge 0\ ,\quad t\ge 0\ .$$

Since $\coth|_{\mathbb{R}^+}$ is bounded from below by 1 and $\tanh(z)\nearrow 1$ for $z\nearrow\infty$, the assertion is a consequence of Lemma 2.13 with

$$c_0:=\frac{2\varphi_*\gamma_*}{\varphi^*\|K\|_\infty(m_c+2)}\ .$$

□

For certain collision kernels, the assumptions on the breakage frequency can be weakened as follows:

Theorem 2.19. *Suppose that* $0<\varphi_*\le\varphi(v)\le\varphi^*<\infty$ *for* $v\in L_1^+$*. Also assume that there are* $\gamma_*,K^*>0$ *such that*

$$\int_0^y\Big(1-\frac{y'}{y}\Big)\gamma(y,y')\ dy'\ge\gamma_*y\ ,\quad a.a.\ y\in Y\ , \tag{2.18}$$

and

$$K(y,y') \le K^*(y+y') \ , \quad a.a.\ (y,y') \in Y \times Y \ .$$

Then there exists $c_0 > 0$, depending only on φ and the kernels, such that

$$\liminf_{t \nearrow \infty} |u(t;u^0)|_1 \ge c_0 \ , \quad u^0 \in L_1^+ \setminus \{0\} \ .$$

Proof. Theorem 2.10 gives $J(u^0) = \mathbb{R}^+$ and thus, due to Lemma 2.7,

$$\dot{M}_0(t) \ge \varphi_* \gamma_* M_1(0) - \varphi^* K^* (m_c + 2) M_1(0) M_0(t) \ , \quad t \ge 0 \ ,$$

from which

$$M_0(t) \ge (M_0(0) - c_0) e^{-\varphi^* K^*(m_c+2) M_1(0) t} + c_0 \ , \quad t \ge 0 \ ,$$

follows with

$$c_0 := \frac{\varphi_* \gamma_*}{\varphi^* K^* (m_c + 2)} \ .$$

□

Corollary 2.20. *If the hypotheses of Theorem 2.18 or of Theorem 2.19 hold, then the trivial solution is not stable for the semiflow on L_1^+.*

Examples 2.21. To illustrate our preceding statements, we consider now some special kernels. We take spontaneous breakage kernels of the form

$$\gamma(y,y') := a(y) b(y,y') \ , \quad 0 < y' < y \le y_0 \ ,$$

where $a(y)$ is the rate at which a droplet of mass y breaks and $b(y,y')$ represents the distribution of fragments formed from a splitting droplet of mass y. Conservation of mass leads to the normalization

$$\int_0^y y' b(y,y') \, dy' = y \ , \quad y \in Y \ . \tag{2.19}$$

Moreover, the quantity

$$\nu(y) := \int_0^y b(y,y') \, dy' \ , \quad y \in Y \ ,$$

gives the expected number of droplets when y breaks. Thus $\nu(y) \ge 2$ if $a(y) > 0$. The case where $a(y)$ has no zeros corresponds to *complete breakage*. If binary breakage is considered, i.e.

$$b(y,y') = b(y, y-y') \ , \quad 0 < y' < y \le y_0 \ , \tag{2.20}$$

then (2.19) implies $\nu(y) = 2,\ y \in Y$.

(I) Consider the case of *limited breakage* (cf. [**65**]) which simply means that there exists a stable droplet size $y_s \in (0, y_0)$ [1], depending mainly on impeller diameter and speed, such that droplets which are smaller than y_s have a zero breakage rate, that is,

$$a(y) = 0 \ , \quad 0 < y \le y_s \ .$$

[1] In some settings (see [**65**]) the stable droplet size can be characterized by

$$y_s = c D^3 (We)^{-1.8}$$

where c is a constant, $We = \omega^2 D^3 \varrho / \sigma$ represents the Weber number and σ and ϱ are the surface tension and the density of the dispersed phase, respectively, ω is the impeller speed and D denotes the impeller diameter.

Then

$$\int_0^y \Big(1-\frac{y'}{y}\Big)\gamma(y,y')\,dy' = a(y)\big(\nu(y)-1\big) \le \frac{\|a\|_\infty\,\|\nu\|_\infty}{y_s}y\ ,\quad y\in Y\ ,$$

for $y \in Y$ provided that a and ν are bounded. Thus (2.12) holds.

(II) Suppose *complete breakage* in a strong form such that

$$a(y) \ge \underline{a}y^k\ ,\quad y\in Y\ ,$$

for $k=0$ or $k=1$ and some $\underline{a}>0$. Since in this case $\nu(y)\ge 2$ and

$$\int_0^y \Big(1-\frac{y'}{y}\Big)\gamma(y,y')\,dy' \ge \underline{a}y^k\ ,\quad y\in Y\ ,$$

the estimates (2.17) or (2.18) are valid.

(III) A *power-law breakup* (see e.g. [**47**], [**65**], or [**67**]) is of the form

$$a(y) := hy^\alpha\ ,\quad b(y,y') := (\zeta+2)y^{-(1+\zeta)}(y')^\zeta\ ,\quad 0<y'<y\le y_0\ ,$$

with $-1<\zeta\le 0$ and $h>0$. In view of (2.20), binary breakage corresponds to $\zeta=0$. The underlying idea is that if droplets are assumed to be spherical, the mass y of a droplet is proportional to d^3 where d denotes its diameter. Accordingly, if the mechanism of breakage is independent of the droplet involved or depends either on the diameter itself, or on the surface area, or on the volume of the droplet, α is given by $0, 1/3, 2/3$, or 1, and, analogously, for ζ. Moreover, we have

$$\nu(y) = \frac{\zeta+2}{\zeta+1}\ ,\quad y\in Y\ ,$$

and

$$\int_0^y \Big(1-\frac{y'}{y}\Big)\gamma(y,y')\,dy' = \frac{h}{1+\zeta}y^\alpha\ ,\quad y\in Y\ .$$

If we also suppose that the shattering kernel β_c satisfies a power-law breakup, i.e.

$$\beta_c(y,y') := (\xi+2)y^{-(1+\xi)}(y')^\xi\ ,\quad 0<y'<y\le y_0\ ,$$

for some $0\ge\xi>-1$, then

$$\int_0^y \beta_c(y,y')\,dy' = \frac{\xi+2}{\xi+1}\ ,\quad y\in Y\ ,$$

and Theorem 2.10 implies that the solution exists globally provided there exists some constant $z_0\in Y$ for which

$$\frac{P(y,y')}{Q(y,y')} \ge \frac{-\xi}{1+\xi}\ ,\quad 0<y+y'\le z_0\ . \tag{2.21}$$

This means that coalescence dominates shattering for small droplets, and this is always fulfilled if either $\xi=0$ (binary shattering) or $Q\equiv 0$ (no shattering). Moreover, if there is some $\varepsilon>0$ with

$$\frac{P(y,y')-\varepsilon}{Q(y,y')} \ge \frac{-\xi}{1+\xi}\ ,\quad 0<y+y'\le z_0\ , \tag{2.22}$$

then (2.13) of Theorem 2.14 is satisfied. Assuming (2.21) to be true and taking coalescence kernels of the form

$$K(y,y') := A + B(y+y')^\sigma + C(yy')^\tau\ ,\quad y,y'\in Y\ , \tag{2.23}$$

with $A, B, C \geq 0$ and $\sigma, \tau \geq 0$, we can distinguish the following cases:

(i) If $\alpha = 0$, which means that the breakage rate does not depend on the droplet size, then Theorem 2.18 implies that the trivial solution $u \equiv 0$ is unstable. Furthermore, if additionally $A > 0$ and (2.22) holds, then, for any initial distribution $u^0 \in L_1^+$, the total number of droplets remains bounded thanks to Theorem 2.14.

(ii) If $\alpha \in (0, 1], A = 0$, and $\sigma, \tau \geq 1$, then the trivial solution is also unstable since Theorem 2.19 holds.

(iii) If $\alpha > 0$, $A > 0$, and (2.22) is valid, then we have stability of $u \equiv 0$ and, given any initial distribution $u^0 \in L_1^+$, the total number of droplets has an upper bound due to Theorem 2.14.

(IV) Consider the case of *parabolic breakup* (cf. [**67**]) meaning that

$$a(y) := hy^\eta \ , \quad b(y, y') := (\omega + 2)(\omega + 3)y^{-(\omega+2)}(y')^\omega(y - y') \ ,$$

with $h > 0$, $\eta \geq 0$, and $1 \geq \omega > -1$. Here, $\omega = 1$ amounts to binary breakage. The expected number of fragments formed by rupture in this case is

$$\nu(y) = \frac{\omega + 3}{\omega + 1} \ , \quad y \in Y \ .$$

Defining K by (2.23) and putting $P \equiv 1$, we can distinguish the same cases as done in (III) since

$$\int_0^y \left(1 - \frac{y'}{y}\right)\gamma(y, y') \ dy' = \frac{2h}{\omega + 1} y^\eta \ , \quad y \in Y \ .$$

3. Long-Time Behaviour

After having provided sufficient conditions for global solutions of Problem $(*)$, we study in this chapter long-time behaviour of these solutions. In section 3.1 it will be shown that they converge weakly to some ω-limit set. This will be done under rather restrictive assumptions on the collision frequency, but without use of binary breakage, in contrast to the subsequent section 3.2. The latter is devoted to examine asymptotic stability of equilibria for kernels satisfying the detailed balance condition in the case of binary breakage.

3.1. Relatively Weakly Compact Orbits

It is the purpose here to prove that under suitable assumptions the orbits of the semiflow

$$\big[(t,u^0)\mapsto u(t;u^0)\big]$$

are relatively compact in the weak topology of L_1, which has similar implications for the motion through $u^0\in L_1^+$ and its weak ω-limit set $\omega(u^0)$ as in the case of metric spaces. For instance, $\omega(u^0)$ is then nonempty and invariant.

We denote by $L_{1,\mathrm{w}}$ the usual space L_1 endowed with its weak topology and by $L_{1,\mathrm{w}}^+$ its positive cone. We assume that φ and the kernels $\gamma, \beta_c, \beta_s, K, P$, and Q satisfy Hypotheses $(H_1)-(H_5)$ and that the following additional hypotheses are valid:

(H_6) There exist $\varphi_*,\varphi^*>0$ with $0<\varphi_*\le\varphi(v)\le\varphi^*<\infty$ for $v\in L_1^+$;

(H_7) for each $\varepsilon>0$ there exists $\delta:=\delta(\varepsilon)>0$ such that for any measurable subset A of Y with measure $|A|\le\delta$ it holds

$$\int_0^y \chi_A(y')\gamma(y,y')\,dy'\le\varepsilon\ ,\quad \text{a.a. } y\in Y\ ,$$

and such that

$$Q(y,y')\int_0^{y+y'}\chi_A(y'')\beta_c(y+y',y'')\,dy''\le\varepsilon\ \ ,\quad \text{a.a. } (y,y') \text{ with } y+y'\in Y\ ,\tag{3.1}$$

and

$$\int_0^{y_0}\chi_A(y'')\beta_s(y+y',y'')\,dy''\le\varepsilon\ ,\quad \text{a.a. } (y,y') \text{ with } y+y'\in(y_0,2y_0]\ ;$$

(H_8) there exist $K_*,K^*>0$ with

$$0<K_*\le K(y,y')\le K^*<\infty\ ,\quad \text{a.a. } (y,y')\in Y\times Y\ ;$$

(H_9) there exists $\varepsilon_0>0$ with

$$\varepsilon_0\le P(y,y')+Q(y,y')\le 1\ ,\quad \text{a.a. } (y,y')\in Y\times Y\ ;$$

(H_{10}) $q:=\varphi_*K_*\varepsilon_0-\frac{1}{2}\varphi^*K^*>0.$

A possible choice of the kernels are given in Examples 2.21. For instance, define

$$\gamma(y,y'):=h(\zeta+2)y^{\alpha-\zeta-1}(y')^{\zeta}\ ,\quad 0<y'<y\le y_0\ ,$$

for some $h > 0$, $0 \geq \zeta > -1$, and $\alpha \geq 1 + \zeta$. Moreover, assume that small droplets have shattering probability zero, that is, there exists some $z_0 \in Y$ with $Q(y,y') = 0$ for $0 < y + y' \leq z_0$. Put

$$\beta_c(y,y') := (\xi+2)y^{-(1+\xi)}(y')^{\xi} \ , \quad 0 < y' < y \leq y_0 \ ,$$

as well as

$$\beta_s(y,y') := (\nu+2)y_0^{-(2+\nu)}y(y')^{\nu} \ , \quad 0 < y' \leq y_0 < y \leq 2y_0 \ ,$$

for some $0 \geq \xi$, $\nu > -1$. Then Hypothesis (H_7) is easily verified.
Obviously, Hypothesis (H_{10}) is the most restrictive one. It implies that the collision frequency K has a rather small range, i.e., it is almost constant. Nevertheless, kernels of the form

$$K(y,y') := A + B(yy')^{\sigma} + C(y+y')^{\tau} \ , \quad \sigma\, , \tau \geq 0 \ ,$$

satisfy Hypothesis (H_{10}) provided A is large. Furthermore, (H_{10}) also implies $\varepsilon_0 > \frac{1}{2}$ so that occurrence of two grazing droplets has a probability of less than $1 - \varepsilon_0 \in [0, \frac{1}{2})$ in view of Hypothesis (H_9). Also observe that (3.1) yields that $Q(y,y')$ tends to zero if y and y' do so. Indeed, since, for physical reasons, the expected number of daughter droplets resulting from shattering is not less than 2, that is,

$$\int_0^{y+y'} \beta_c(y+y',y'')\, dy'' \geq 2 \ , \quad \text{a.a. } (y,y') \text{ with } y+y' \in Y \ ,$$

(3.1) implies that for each $\varepsilon > 0$ there exists some $\delta > 0$ with

$$2Q(y,y') \leq \varepsilon \ , \quad \text{a.a. } (y,y') \text{ with } y+y' \in (0,\delta] \ .$$

Observe that Hypotheses (H_7) and (H_9) entail

$$\int_0^{y+y'} \beta_c(y+y',y'')\, dy'' \leq 2 + \frac{P(y,y') - \varepsilon_0/2}{Q(y,y')} \ , \quad \text{a.a. } (y,y') \text{ with } 0 < y+y' \leq \delta\Big(\frac{\varepsilon_0}{2}\Big) \ ,$$

for some $\delta(\frac{\varepsilon_0}{2}) > 0$. Therefore, the previous chapter and in particular Theorem 2.14 provide that for any $u^0 \in L_1^+$ the solution $u = u(\cdot; u^0)$ belongs to $C^1(\mathbb{R}^+, L_1^+)$, and there exists some constant $c(|u^0|_1) > 0$ with

$$|u(t)|_1 \leq c(|u^0|_1) \ , \quad t \geq 0 \ . \tag{3.2}$$

We then define the positive orbit of the motion through $u^0 \in L_1^+$ by

$$\gamma^+(u^0) := \big\{u(t;u^0) \ ; \ t \geq 0\big\} \ ,$$

and as before we put $\varphi(t) := \varphi\big(u(t)\big)$, $t \geq 0$.
In order to prove relative weak compactness of the orbits in L_1, we adapt an idea used in [**58**] for constant kernels.

Theorem 3.1. *For each $u^0 \in L_1^+$ the positive orbit $\gamma^+(u^0)$ is relatively compact in $L_{1,\mathrm{w}}^+$.*

Proof. We may assume that $u^0 \neq 0$. For fixed $\delta > 0$ define $Z_\delta \in C(\mathbb{R}^+)$ by

$$Z_\delta(t) := \sup \int_B u(t,y)\, dy \ , \quad t \geq 0 \ ,$$

where the supremum is taken over all measurable subsets B of Y having measure $|B| \le \delta$. Moreover, put

$$\begin{aligned}\mu(\delta) := &\sup_{|B|\le\delta}\ \operatorname*{ess-sup}_{y\in Y} \int_0^y \chi_B(y')\gamma(y,y')\ dy' \\ &+ \sup_{|B|\le\delta}\ \operatorname*{ess-sup}_{y+y'\in Y} Q(y,y')\int_0^{y+y'} \chi_B(y'')\beta_c(y+y',y'')\ dy'' \\ &+ \sup_{|B|\le\delta}\ \operatorname*{ess-sup}_{y+y'\in(y_0,2y_0]} \int_0^{y_0} \chi_B(y'')\beta_s(y+y',y'')\ dy''\ .\end{aligned}$$

Choose any measurable subset A of Y with measure $|A| \le \delta$ and define

$$N_A(t) := \int_A u(t,y)\ dy\ ,\quad t\ge 0\ ,$$

as well as

$$N(t) := \int_Y u(t,y)\ dy\ ,\quad t\ge 0\ .$$

Since $u = u(\cdot;u^0) \in C^1(\mathbb{R}^+, L_1^+)$ we obtain from Lemma 2.7, Hypotheses $(H_6)-(H_{10})$, and (3.2) that

$$\begin{aligned}\frac{d}{dt}N_A(t) \le\ &\varphi(t)\int_0^{y_0}\int_0^y \chi_A(y')\gamma(y,y')\ dy'\ u(t,y)\ dy \\ &+\frac{1}{2}\varphi(t)\int_0^{y_0}\int_0^{y_0-y} \chi_{-y+A}(y')P(y,y')K(y,y')u(t,y)u(t,y')\ dy'dy \\ &-\frac{1}{2}\varphi(t)\int_0^{y_0}\int_0^{y_0-y} \big[\chi_A(y)+\chi_A(y')\big]\big[P(y,y')+Q(y,y')\big] \\ &\qquad\qquad K(y,y')u(t,y)u(t,y')\ dy'dy \\ &+\frac{1}{2}\varphi(t)\int_0^{y_0}\int_0^{y_0-y} Q(y,y')\int_0^{y+y'} \chi_A(y'')\beta_c(y+y',y'')\ dy'' \\ &\qquad\qquad K(y,y')u(t,y)u(t,y')\ dy'dy \\ &+\frac{1}{2}\varphi(t)\int_0^{y_0}\int_{y_0-y}^{y_0}\int_0^{y_0} \chi_A(y'')\beta_s(y+y',y'')\ dy'' \\ &\qquad\qquad K(y,y')u(t,y)u(t,y')\ dy'dy \\ &-\frac{1}{2}\varphi(t)\int_0^{y_0}\int_{y_0-y}^{y_0} \big[\chi_A(y)+\chi_A(y')\big]K(y,y')u(t,y)u(t,y')\ dy'dy \\ \le\ & c(u^0)\mu(\delta)+\frac{1}{2}\varphi^*K^*N(t)Z_\delta(t)-\varphi_*\varepsilon_0K_*N(t)N_A(t)\ ,\end{aligned}$$

where we additionally used that Lebesgue's measure is invariant with respect to translations. Defining

$$d(t) := e^{\varphi_*\varepsilon_0K_*\int_0^t N(s)\,ds}\ ,\quad t\ge 0\ ,$$

it follows

$$\frac{d}{dt}\big[N_A(t)d(t)\big] \le c(u^0)\mu(\delta)d(t)+\frac{1}{2}\varphi^*K^*N(t)d(t)Z_\delta(t)\ ,\quad t\ge 0\ . \tag{3.3}$$

Now put

$$v_\delta(t) := Z_\delta(t)d(t)\ ,\quad t\ge 0\ ,$$

and

$$w_\delta(t) := Z_\delta(0) + c(u^0)\mu(\delta)\int_0^t d(s)\, ds\ , \quad t\geq 0\ .$$

Integrating (3.3) from 0 to t and taking then the suprema over all measurable subsets A of Y with measure $|A|\leq\delta$, we deduce

$$v_\delta(t)\leq w_\delta(t)+\frac{1}{2}\varphi^* K^*\int_0^t N(s)v_\delta(s)\, ds\ , \quad t\geq 0\ ,$$

so that by the Gronwall-Bellman inequality (see [**50**, Thm.1.3.2]) for $t\geq 0$

$$\begin{aligned} v_\delta(t) &\leq w_\delta(t)+\frac{1}{2}\varphi^*K^*\int_0^t w_\delta(s)N(s)e^{\frac{1}{2}\varphi^*K^*\int_s^t N(\sigma)\,d\sigma}\, ds \\ &= w_\delta(t) - w_\delta(s)e^{\frac{1}{2}\varphi^*K^*\int_s^t N(\sigma)\,d\sigma}\Big|_{s=0}^{s=t} + \int_0^t \dot w_\delta(s)e^{\frac{1}{2}\varphi^*K^*\int_s^t N(\sigma)\,d\sigma}\, ds \\ &= Z_\delta(0)e^{\frac{1}{2}\varphi^*K^*\int_0^t N(\sigma)\,d\sigma} + c(u^0)\mu(\delta)\int_0^t d(s)e^{\frac{1}{2}\varphi^*K^*\int_s^t N(\sigma)\,d\sigma}\, ds\ . \end{aligned}$$

Multiplying both sides of the above inequality by $d(t)^{-1}$ and taking into account

$$N(t)\geq\frac{1}{y_0}\int_0^{y_0} yu^0(y)\, dy =: \varrho(u^0)\ , \quad t\geq 0\ ,$$

we conclude due to Hypothesis (H_{10}) that for $t\geq 0$

$$\begin{aligned} Z_\delta(t) &\leq Z_\delta(0)e^{-q\int_0^t N(\sigma)\,d\sigma} + c(u^0)\mu(\delta)\int_0^t e^{-q\int_s^t N(\sigma)\,d\sigma}\, ds \\ &\leq Z_\delta(0) + c(u^0)\big(q\varrho(u^0)\big)^{-1}\mu(\delta)\ . \end{aligned}$$

Obviously, the right hand side tends to zero as δ tends to zero. Hence, for given $\varepsilon>0$ we can choose $\delta>0$ small enough to obtain

$$\int_A u(t,y)\, dy\leq Z_\delta(t)\leq\varepsilon\ , \quad t\geq 0\ , \tag{3.4}$$

for each measurable subset A of Y with $|A|\leq\delta$. From (3.2), (3.4), and the Dunford-Pettis theorem (see [**25**, Thm.4.21.2]) the assertion follows. □

Define for $u^0\in L_1^+$ the weak ω-limit set by

$$\omega(u^0) := \big\{v\in L_1\ ;\ \text{there exists a sequence } t_n\to\infty \text{ with } u(t_n;u^0)\to v \text{ in } L_{1,\mathrm{w}}\big\}\ .$$

Theorem 3.2. *Let $u^0\in L_1^+$. Then the weak ω-limit set $\omega(u^0)\subset L_1^+$ is nonempty, relatively weakly compact in L_1, and if $v\in\omega(u^0)$, then*

$$\int_0^{y_0} yv(y)\, dy = \int_0^{y_0} yu^0(y)\, dy\ . \tag{3.5}$$

Moreover, it holds $u(t;u^0)\to\omega(u^0)$ in $L_{1,\mathrm{w}}$ as $t\to\infty$.

Proof. Let $t_n\to\infty$ be arbitrary. Since $\gamma^+(u^0)$ is relatively weakly compact in L_1 by Theorem 3.1, there exists a subsequence $(t_{n'})$ and $v\in L_1$ such that

$$u(t_{n'};u^0)\to v \quad \text{in} \quad L_{1,\mathrm{w}}\ . \tag{3.6}$$

Thus $\omega(u^0)$ is nonempty. Next, let U be an open neighbourhood of $\omega(u^0)$ in $L_{1,\mathrm{w}}$ and assume that there exists a sequence $t_n\to\infty$ such that $u(t_n;u^0)$ belongs to the compact

set $U^c \cap cl_{L_{1,\mathrm{w}}}\gamma^+(u^0)$. Hence, there exists a subsequence $(t_{n'})$ and $v \in L_1$ such that (3.6) holds. But then $v \in \omega(u^0) \subset U$ contradicting the fact that the weak limit of the sequence $\big(u(t_{n'};u^0)\big) \subset U^c$ has to belong to U^c. This implies that the assumption is false and therefore $u(t;u^0) \to \omega(u^0)$ in $L_{1,\mathrm{w}}$ as $t \to \infty$.
Since the map $y \mapsto y$ belongs to $L_\infty(Y)$, (3.5) is true. L_1^+ being weakly closed, $\omega(u^0)$ is indeed contained in L_1^+.
Finally, the Dunford-Pettis theorem and Theorem 3.1 entail that $\omega(u^0)$ is relatively weakly compact if one observes that $u(t_n;u^0) \to v$ in $L_{1,\mathrm{w}}$ implies

$$\int_A u(t_n;u^0)(y)\ dy \to \int_A v(y)\ dy$$

for any measurable subset A of Y. □

In order to prove that the weak ω-limit set is invariant, we need to know that the solution $u(\cdot;u^0)$ depends on u^0 with respect to the weak topology of L_1. This is the purpose of the next proposition.

Proposition 3.3. *In addition to the Hypotheses $(H_1)-(H_{10})$ assume that $\varphi : L_{1,\mathrm{w}} \to \mathbb{R}^+$ is sequentially continuous and let $u_n^0 \to u^0$ in $L_{1,\mathrm{w}}^+$. Then, for each $T > 0$, it holds*

$$u(\cdot;u_n^0) \to u(\cdot;u^0) \quad in \quad C([0,T],L_{1,\mathrm{w}})\ .$$

Proof. Put $u_n(t) := u(t;u_n^0)$ for $t \geq 0$. First we show that the set $\big\{u_n\,;\,n\in\mathbb{N}\big\}$ is relatively sequentially compact in the locally convex space $C([0,T],L_{1,\mathrm{w}})$. Since $(u_n^0)_{n\in\mathbb{N}}$ is bounded in L_1, (3.2) guarantees that there exists some $R_0 > 0$ such that

$$|u_n(t)|_1 \leq R_0\ , \quad t \geq 0\ , \quad n \in \mathbb{N}\ . \tag{3.7}$$

For any given $\delta > 0$ define

$$Z_\delta^n(t) := \sup \int_B u_n(t,y)\ dy\ , \quad t \geq 0\ ,$$

where the supremum is taken over all measurable subsets B of Y having measure $|B| \leq \delta$. Analogously to the proof of Theorem 3.1 it follows then with the aid of Hypotheses $(H_1)-(H_{10})$ and (3.7) that for fixed $T > 0$

$$Z_\delta^n(t) \leq c(T,R_0)\big(\mu(\delta) + Z_\delta^n(0)\big)\ , \quad 0 \leq t \leq T\ ,$$

where $c(T,R_0) > 0$ is independent of $n \in \mathbb{N}$ and where $\mu(\delta)$ is defined as in the proof of Theorem 3.1 (observe that for this estimate the lower bound K_* in Hypothesis (H_8) is not needed). The Dunford-Pettis theorem implies that $Z_\delta^n(0) \to 0$ uniformly with respect to $n \in \mathbb{N}$ as $\delta \to 0$. Therefore, for any given $\varepsilon > 0$ there exists $\delta > 0$ such that for each measurable subset A of Y with measure $|A| \leq \delta$ we have

$$\int_A u_n(t,y)\ dy \leq Z_\delta^n(t) \leq \varepsilon\ , \quad 0 \leq t \leq T\ , \quad n \in \mathbb{N}\ .$$

This and (3.7) entail then that the set $\big\{u_n(t)\,;\,n\in\mathbb{N}\big\}$ is for each $t \in [0,T]$ relatively weakly compact in L_1, again due to the Dunford-Pettis theorem. Moreover, since

$$u_n(t) = u_n^0 + \int_0^t \varphi\big(u_n(\sigma)\big)L[u_n(\sigma)]\ d\sigma\ , \quad t \geq 0\ , \tag{3.8}$$

we deduce from (3.7) and Lemma 2.1 that there exists $c(R_0) > 0$ with

$$|u_n(t) - u_n(s)|_1 \leq \varphi^* \Big| \int_s^t |L[u_n(\sigma)]|_1 \, d\sigma \Big| \leq c(R_0)|t-s| \tag{3.9}$$

for $0 \leq s,t \leq T$ and $n \in \mathbb{N}$, i.e., the set $\big\{u_n|_{[0,T]} \, ; \, n \in \mathbb{N}\big\}$ is equicontinuous with respect to the L_1-topology and thus also with respect to the $L_{1,\mathrm{w}}$-topology. By a version of the Arzelà-Ascoli theorem (see [**68**, Thm.1.3.2]) we can extract a subsequence (n') of (n) such that

$$u_{n'} = u(\cdot; u^0_{n'}) \to v \quad \text{in} \quad C([0,T], L_{1,\mathrm{w}}) \tag{3.10}$$

for some $v \in C([0,T], L_{1,\mathrm{w}})$. Then (3.9) and (3.10) imply $v \in C^{1-}([0,T], L_1^+)$. In the appendix to this chapter (see Corollary A.5) it is shown that $L[\cdot]$ is weakly sequentially continuous so that by (3.10)

$$L[u_{n'}(\sigma)] \to L[v(\sigma)] \quad \text{in} \quad L_{1,\mathrm{w}} \; , \quad 0 \leq \sigma \leq T \; . \tag{3.11}$$

Now let $0 \leq t \leq T$ and $f \in L_\infty(Y)$ be arbitrary. Then

$$\begin{aligned} \Big| \int_0^{y_0} & f(y) \int_0^t \varphi\big(u_{n'}(\sigma)\big) L[u_{n'}(\sigma)] \, d\sigma dy - \int_0^{y_0} f(y) \int_0^t \varphi\big(v(\sigma)\big) L[v(\sigma)] \, d\sigma dy \Big| \\ &\leq \varphi^* \int_0^t \Big| \int_0^{y_0} f(y) \big\{ L[u_{n'}(\sigma)](y) - L[v(\sigma)](y) \big\} \, dy \Big| d\sigma \\ &\quad + \int_0^t \big| \varphi\big(u_{n'}(\sigma)\big) - \varphi\big(v(\sigma)\big) \big| \int_0^{y_0} \big| f(y) L[v(\sigma)](y) \big| \, dy d\sigma \; . \end{aligned}$$

Using (3.11), (3.10), (3.7), Lemma 2.1, and the assumption that φ is weakly sequentially continuous, we may apply Lebesgue's theorem to deduce that the right-hand side of the above inequality tends to zero as $n' \to \infty$. Since $f \in L_\infty(Y)$ being arbitrary, it follows

$$\int_0^t \varphi\big(u_{n'}(\sigma)\big) L[u_{n'}(\sigma)] \, d\sigma \longrightarrow \int_0^t \varphi\big(v(\sigma)\big) L[v(\sigma)] \, d\sigma \quad \text{in} \quad L_{1,\mathrm{w}} \; , \quad 0 \leq t \leq T \; .$$

Because weak limits are unique, we obtain from (3.8) that $v \in C([0,T], L_1^+)$ is a mild solution of Problem $(*)$ with initial value u^0. Hence $v = u(\cdot; u^0)|_{[0,T]}$ since mild solutions are unique and therefore

$$u_{n'} \to u(\cdot; u^0) \quad \text{in} \quad C([0,T], L_{1,\mathrm{w}}) \; .$$

Because this limit is independent of the subsequence (n'), the assertion follows. □

Theorem 3.4. *In addition to the Hypotheses $(H_1) - (H_{10})$ assume that $\varphi : L_{1,\mathrm{w}} \to \mathbb{R}^+$ is sequentially continuous. Then, for each $u^0 \in L_1^+$, the weak ω-limit set $\omega(u^0)$ is invariant, that is, $u\big(t; \omega(u^0)\big) = \omega(u^0)$ for $t \geq 0$.*

PROOF. Let $t_n \to \infty$ and

$$u(t_n; u^0) \to v \quad \text{in} \quad L_{1,\mathrm{w}} \; . \tag{3.12}$$

Fix $t \geq 0$ arbitrarily. Since $v \in L_1^+$, the semiflow property and Proposition 3.3 entail that

$$u(t + t_n; u^0) = u\big(t; u(t_n; u^0)\big) \to u(t; v) \quad \text{in} \quad L_{1,\mathrm{w}} \; ,$$

and hence $u(t;v) \in \omega(u^0)$. On the other hand, by Theorem 3.1 we can extract a subsequence (n') and find $w \in L_1^+$ such that $u(t_{n'} - t; u^0) \to w$ in $L_{1,\mathrm{w}}$. Thus, $w \in \omega(u^0)$ and, by Proposition 3.3, $u(t_{n'}; u^0) \to u(t; w)$ in $L_{1,\mathrm{w}}$. We then deduce $v = u(t;w) \in u\big(t; \omega(u^0)\big)$ from (3.12). □

Remark 3.5. For instance, if $\varphi(u)$ is of the form

$$\varphi(u) = \Phi\Big(\int_0^{y_0} u(y)\ dy\ , \int_0^{y_0} y^{2/3} u(y)\ dy \Big)$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is a given Lipschitz continuous function which is bounded below and above by some positive constants (see the explanation in the introduction on page 6f), then φ fulfills Hypotheses (H_1) and (H_6) and is weakly sequentially continuous.

3.2. Trend to Equilibrium

It is the aim of this section to investigate a particular case of the binary coalescence-breakage equations, namely when the kernels satisfy the detailed balance condition (see Hypothesis (H_{14}) below), which amounts to assume that the processes under consideration are somehow reversible. This condition not only ensures existence of equilibria but also provides a Lyapunov function. Inspired by the pioneering works of [**1**] for $K \equiv \gamma \equiv 2$ in the continuous case and [**16**] for the (discrete) Becker-Döring equations, this Lyapunov function has been widely used both for the discrete (for instance, see [**19**], [**20**]) and for the continuous (see [**39**], [**58**]) coagulation-fragmentation equations to prove convergence of the solutions towards equilibria in appropriate topologies (for results concerning existence of and convergence towards equilibria not assuming the detailed balance condition, we refer to [**60**]).
We extend this detailed balance condition to include shattering and scattering and use similar arguments as in [**39**] to prove that the solution tends towards the unique equilibrium with mass equal to that of the initial distribution. This convergence is true with respect to the weak topology of L_1 but can be improved to convergence in the strong L_1-topology, assuming slightly stricter conditions on the breakage kernel. Further, we derive stability of these equilibria in a stronger topology than the L_1-topology.

Again, we assume Hypotheses $(H_1) - (H_5)$ to be satisfied and suppose that the following additional hypotheses hold:

(H_{11}) There exists $\varphi^* > 0$ with $0 < \varphi(v) \le \varphi^*$ for $v \in L_1^+$, and $\varphi : L_{1,\mathrm{w}} \to \mathbb{R}$ is sequentially continuous;

(H_{12}) the kernels γ, β_c, and β_s satisfy

$$\gamma(y,y') = \gamma(y, y-y')\ , \quad \beta_c(y,y') = \beta_c(y,y-y')\ , \quad 0 < y' < y < y_0\ ,$$

and

$$\beta_s(y,y') = \beta_s(y,y-y')\ , \quad 0 < y - y_0 < y' < y_0\ ,$$

as well as

$$\beta_s(y,y') = 0\ , \quad 0 < y' < y - y_0\ ; \tag{3.13}$$

(H_{13}) $P \cdot K > 0$ a.e.;

(H_{14}) there exists $H \in L_1^+$ with $h_0 := \text{ess-inf}\ H > 0$ and

(i) for $0 < y + y' < y_0$ it holds

$$\gamma(y+y',y)H(y+y') = P(y,y')K(y,y')H(y)H(y')\ ,$$

(ii) for $0 < y+y',\ y+y'' < y_0$ it holds

$$\begin{aligned}\beta_c(y,y')Q(y'',y-y'')&K(y'',y-y'')H(y'')H(y-y'')\\ &= \beta_c(y,y'')Q(y',y-y')K(y',y-y')H(y')H(y-y')\ ,\end{aligned}$$

(iii) for $0 < y - y_0 < y'$, $y'' < y_0$ it holds

$$\beta_s(y,y')K(y'',y-y'')H(y'')H(y-y'') = \beta_s(y,y'')K(y',y-y')H(y')H(y-y') \ .$$

We refer to Examples 3.17 for kernels satisfying the hypotheses above. Hypothesis (H_{11}) is introduced in order to exclude possible roots of φ as equilibria. Hypothesis (H_{12}) means binary breakage as being explained in the introduction. Note that, due to (3.13), the scattering operator is given by

$$\begin{aligned} L_s[v](y) &:= L_s^1[v](y) - L_s^2[v](y) \\ &:= \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} K(y'',y'-y'')\beta_s(y',y)v(y'')v(y'-y'')\ dy''dy' \\ &\quad - v(y)\int_{y_0-y}^{y_0} K(y,y')v(y')\ dy' \ . \end{aligned}$$

Moreover, (H_{12}) and conservation of mass (see (H_3) and (H_4)) imply that for a.a. (y,y') with $y+y' \in Y$

$$2\int_0^{y+y'} \frac{y''}{y+y'}\beta_c(y+y',y'')\ dy'' = \int_0^{y+y'} \beta_c(y+y',y'')\ dy'' = 2 \ , \tag{3.14}$$

and that for a.a. (y,y') with $y+y' \in (y_0, 2y_0]$

$$2\int_{y+y'-y_0}^{y_0} \frac{y''}{y+y'}\beta_s(y+y',y'')\ dy'' = \int_{y+y'-y_0}^{y_0} \beta_s(y+y',y'')\ dy'' = 2 \ . \tag{3.15}$$

Likewise, for a.a. $y \in Y$

$$\int_0^y \frac{y'}{y}\gamma(y,y')\ dy' = \frac{1}{2}\int_0^y \gamma(y,y')\ dy' \ ,$$

so that the breakage operator L_b takes the form

$$L_b[v](y) := L_b^1[v](y) - L_b^2[v](y) := \int_y^{y_0} \gamma(y',y)v(y')\ dy' - \frac{1}{2}v(y)\int_0^y \gamma(y,y')\ dy' \ .$$

Therefore, according to Theorem 2.10, solutions corresponding to non-negative initial distributions exist globally.

It is easy to check that, due to Hypothesis (H_{14}), the function

$$u_\alpha(y) := H(y)e^{\alpha y} \ , \quad y \in Y \ ,$$

is for each $\alpha \in \mathbb{R}$ an equilibrium of Problem $(*)$, that is,

$$L[u_\alpha] = L_b[u_\alpha] + L_c[u_\alpha] + L_s[u_\alpha] = 0 \ .$$

Also, it is rather more circumstantial than difficult to show *formally* that the map V, defined by

$$V(v) := \int_0^{y_0} \Big\{ v(y)\big[\log\frac{v(y)}{H(y)} - 1\big] + H(y)\Big\}\ dy \ , \quad v \in L_1^+ \ , \tag{3.16}$$

is a Lyapunov function for Problem $(*)$ meaning that V is non-increasing along orbits. To prove it rigorously, much more work is needed. We start with some preliminaries which will be important later for that purpose.

Lemma 3.6. *The map* $V : L_{1,\mathrm{w}}^+ \to \bar{\mathbb{R}}^+$ *is sequentially lower semi-continuous.*

PROOF. Since the map $z \mapsto z \log z$ is convex, it follows that also V is convex. For $b > 0$ define

$$k(a,b) := \begin{cases} a\big(\log \frac{a}{b} - 1\big) + b \ , & a > 0 \ , \\ b \ , & a = 0 \ . \end{cases}$$

Then $k(a,b) \geq k(b,b) = 0$ for $a \geq 0$ and thus $V(w) \geq 0$ for $w \in L_1^+$. Next let $w_n \to w$ in L_1^+. We may extract a subsequence (n') such that

$$\lim_{n'} V(w_{n'}) = \liminf_{n} V(w_n)$$

and such that $w_{n'} \to w$ almost everywhere. But then by Fatou's lemma

$$V(w) \leq \liminf_{n'} \int_0^{y_0} k\big(w_{n'}(y), H(y)\big)\ dy = \liminf_{n} V(w_n) \ .$$

Therefore, $V : L_1^+ \to \bar{\mathbb{R}}^+$ is sequentially lower semi-continuous. Since V is convex, the assertion is a consequence of [**26**, Prop.2.3] and the fact that convex sets are closed iff they are weakly closed. □

Lemma 3.7. *Let* Ω *be a measurable and non-trivial subset of* $\mathbb{R}^m, m \geq 1$*, and define the map* $\mathcal{J} : \mathbb{R}^2 \to \bar{\mathbb{R}}^+$ *by*

$$\mathcal{J}(a,b) := \begin{cases} (a-b)(\log a - \log b) \ , & a,b > 0 \ , \\ 0 \ , & a = b = 0 \ , \\ \infty \ , & else \ . \end{cases} \tag{3.17}$$

Moreover, put

$$J(v,w) := \int_\Omega \mathcal{J}\big(v(z), w(z)\big)\ dz \ , \quad (v,w) \in L_1(\Omega, \mathbb{R}^2) \ .$$

Then $J : L_{1,\mathrm{w}}(\Omega, \mathbb{R}^2) \to \bar{\mathbb{R}}^+$ *is sequentially lower semi-continuous.*

PROOF. It is easy to check that $\mathcal{J}$ is convex and sequentially lower semi-continuous. Hence $J : L_1(\Omega, \mathbb{R}^2) \to \bar{\mathbb{R}}^+$ is convex and sequentially lower semi-continuous due to Fatou's lemma. The assertion follows now from [**26**, Prop.2.3]. □

For the sake of completeness, we prove the following lemmas, even though the statements and their proofs are just slight modifications of those in [**39**].

Lemma 3.8. *Let* $w \in L_1^+$ *be such that* $V(w) < \infty$*. Then, for each* $\alpha \geq e^2$ *and any measurable subset* A *of* Y*, it holds*

$$\int_A w(y)\ dy \leq 2\alpha \int_A H(y)\ dy + \frac{2}{\log \alpha} V(w) \ .$$

PROOF. Fix $\alpha \geq e^2$ and choose any measurable subset A of Y. From the inequality

$$r|\log r| \leq r \log r + \frac{2}{e} \ , \quad r \geq 0 \ , \tag{3.18}$$

it follows

$$\chi_A w \big| \log \frac{w}{H} \big| \leq w \log \frac{w}{H} - w + H + \chi_A w \qquad \text{a.e.} \ . \tag{3.19}$$

By splitting A into $A\cap[w\geq \alpha H]$ and $A\cap[w<\alpha H]$, we deduce

$$\begin{aligned}\int_A w(y)\ dy &\leq \frac{1}{\log\alpha}\int_A w(y)\big|\log\frac{w(y)}{H(y)}\big|\ dy+\alpha\int_A H(y)\ dy\\ &\leq \frac{1}{\log\alpha}V(w)+\frac{1}{\log\alpha}\int_A w(y)\ dy+\alpha\int_A H(y)\ dy\ ,\end{aligned}$$

where the last inequality is due to (3.19). □

Lemma 3.9. *Suppose that $w\in L_1^+$ satisfies for a.a. (y,y') with $0<y+y'\leq y_0$*

$$\gamma(y+y',y)w(y+y')=P(y,y')K(y,y')w(y)w(y')\ . \tag{3.20}$$

Then either $w=0$ a.e. or there exists $\alpha\in\mathbb{R}$ such that $w(y)=H(y)e^{\alpha y}$ for a.a. $y\in Y$.

Proof. Set $v(y):=w(y)/H(y)$ for $y\in(0,y_0]$ and note that $v\in L_1^+$ and

$$v(y+y')=v(y)v(y')\ ,\quad \text{a.a. } (y,y') \text{ with } 0<y+y'\leq y_0\ , \tag{3.21}$$

due to (H_{13}) and (H_{14}). Moreover, for

$$f(y):=\int_0^y v(y')\ dy'\ ,\quad 0\leq y\leq y_0\ ,$$

and for fixed $z\in(0,y_0)$ we have

$$v(y)f(z)=f(y+z)-f(y)\ ,\quad \text{a.a. } y\in(0,y_0-z]\ .$$

Since f is continuous on $[0,y_0]$, we may assume $f(z)>0$, otherwise one easily checks that necessarily $f=0$ and thus $v=0$ a.e.. Put

$$h(y):=\frac{f(y+z)-f(y)}{f(z)}\ ,\quad y\in[0,y_0-z]\ .$$

Then $h=v$ a.e. on $(0,y_0-z]$ and, since $h\in C([0,y_0-z],\mathbb{R}^+)$, we see that h satisfies (3.21) for all $(y,y')\in[0,y_0-z]^2$ with $0\leq y+y'\leq y_0-z$. This and the continuity of h imply $h(y)=e^{\alpha y}$, $y\in[0,y_0-z]$, for some $\alpha\in\mathbb{R}$. Now z can be chosen arbitrarily small so that the assertion readily follows, since α does not depend on z. □

The next proposition shows in fact that the coalescence-breakage equation admits a unique global solution in $L_\infty(Y)$ if all data are bounded. To this end, let us introduce some further notations.
Recall the definition of Δ in Hypothesis (H_2) and put

$$\Sigma:=\big\{(y,y')\in(y_0,2y_0]\times(0,y_0]\ ;\ y-y_0<y'\leq y_0\big\}\ .$$

Assume that the probabilities P and Q both satisfy Hypothesis (H_5). For a symmetric function $\tilde K\in L_\infty^+(Y\times Y,\mathbb{R}^+)$ and for measurable maps $\tilde\gamma,\tilde\beta_c:\Delta\to\mathbb{R}^+$ and $\tilde\beta_s:\Sigma\to\mathbb{R}^+$ define $\tilde L_b:=\tilde L_b(\tilde\gamma)$ as well as $\tilde L_c:=\tilde L_c(\tilde\beta_c,\tilde K)$ and $\tilde L_s:=\tilde L_s(\tilde\beta_s,\tilde K)$ by

$$\begin{aligned}
\tilde{L}_b[w](y) &:= \int_y^{y_0} \tilde{\gamma}(y',y)w(y')\,dy' - \frac{1}{2}w(y)\int_0^y \tilde{\gamma}(y,y')\,dy'\ , \\
\tilde{L}_c[w](y) &:= \frac{1}{2}\int_0^y \tilde{K}(y',y-y')P(y',y-y')w(y')w(y-y')\,dy' \\
&\quad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} \tilde{K}(y'',y'-y'')Q(y'',y'-y'')\tilde{\beta}_c(y',y)w(y'')w(y'-y'')\,dy''dy' \\
&\quad - w(y)\int_0^{y_0-y} \tilde{K}(y,y')P(y,y')w(y')\,dy' \\
&\quad - \frac{1}{2}w(y)\int_0^{y_0-y}\int_0^{y+y'} \tilde{\beta}_c(y+y',y'')\,dy''\ \tilde{K}(y,y')Q(y,y')w(y')\,dy'\ , \\
\tilde{L}_s[w](y) &:= \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} \tilde{\beta}_s(y',y)\tilde{K}(y'',y'-y'')w(y'')w(y'-y'')\,dy''dy' \\
&\quad - \frac{1}{2}w(y)\int_{y_0-y}^{y_0}\int_{y+y'-y_0}^{y_0} \tilde{\beta}_s(y+y',y'')\,dy''\ \tilde{K}(y,y')w(y')\,dy'\ ,
\end{aligned}$$

for $y \in Y$ and $w \in L_1$. Furthermore, define $\tilde{L} := \tilde{L}(\tilde{\gamma},\tilde{\beta}_c,\tilde{\beta}_s,\tilde{K})$ by

$$\tilde{L}[w] := \tilde{L}_b[w] + \tilde{L}_c[w] + \tilde{L}_s[w]\ , \quad w \in L_1\ . \tag{3.22}$$

The additional terms in the above definitions of the fourth integral of $\tilde{L}_c$ and the second integral of $\tilde{L}_s$ guarantee that V is decreasing along orbits of the problem

$$\dot{w} = \varphi(w)\tilde{L}[w]\ , \quad t > 0\ , \qquad w(0) = w^0\ , \tag{3.23}$$

even in the case when $\tilde{\beta}_c$ and $\tilde{\beta}_s$ do not satisfy (3.14) and (3.15), respectively.

In the sequel we denote by $|\cdot|_\infty$ the norm of $L_\infty := L_\infty(Y)$.

Proposition 3.10. *Let $\tilde{\gamma},\tilde{\beta}_c,\tilde{\beta}_s$, and $\tilde{K}$ as above be bounded on their domains. Then, for each $w^0 \in L_\infty^+$, Problem (3.23) admits a unique solution $w := w(\cdot;w^0)$ belonging to $C^1(\mathbb{R}^+,L_\infty^+)$. Moreover, if $w^0 \geq r_0$ a.e. for some $r_0 \in (0,\infty)$ then, for any $T > 0$, there exists $r_T > 0$ such that*

$$w(t) \geq r_T > 0 \quad a.e.\ , \quad 0 \leq t \leq T\ . \tag{3.24}$$

PROOF. First note that the assumptions imposed imply that

$$|\tilde{L}[w]|_\infty \leq c\big(1 + |w|_1\big)|w|_\infty\ , \quad w \in L_\infty\ . \tag{3.25}$$

In particular, the right-hand side of (3.23) is Lipschitz continuous from L_∞ into itself. From this, local existence follows. Positivity and (3.24) is obtained along the lines of the proofs of Theorem 2.4 and Corollary 2.5.
Observe then that

$$\int_0^{y_0} \tilde{L}_b[v](y)\,dy \leq c|v|_1\ , \qquad \int_0^{y_0} \tilde{L}_c[v](y)\,dy \leq 0\ , \qquad \int_0^{y_0} \tilde{L}_s[v](y)\,dy = 0$$

for $v \in L_1^+$. Thus, Gronwall's inequality yields $|w(t)|_1 \leq ce^{ct}$, $t \in J(w^0)$, for some $c := c(w^0) > 0$, where $J(w^0)$ denotes the maximal interval of existence of the solution w. Global existence follows then from (3.25). ◻

For the following proposition — which in fact guarantees that the 'entropy' V is non-increasing along orbits — we have to introduce some further notations. Define the sets

$$\mathcal{E} := \big\{(y,y') \in Y^2 \ ;\ 0 < y+y' < y_0\big\}\ ,$$
$$\mathcal{W} := \big\{(y,y',y'') \in Y^3 \ ;\ 0 < y'' < y+y' < y_0\big\}\ ,$$
$$\mathcal{S} := \big\{(y,y',y'') \in Y^3 \ ;\ y_0 - y'' < y+y'-y'' < y_0\big\}\ ,$$

as well as for $n \geq 1$

$$A_n := \big\{(y,y') \in \mathcal{E} \ ;\ \gamma(y+y',y) \leq n\big\}\ ,$$
$$B_n := \big\{(y,y') \in \mathcal{E} \ ;\ \beta_c(y+y',y) \leq n\big\}\ ,$$
$$C_n := \big\{(y,y') \in Y^2 \setminus \mathcal{E} \ ;\ \beta_s(y+y',y) \leq n\big\}\ .$$

Moreover, truncate the kernels according to

$$\gamma_n(y+y',y) := \begin{cases} \gamma(y+y',y)\ , & (y,y') \in A_n \cap B_n\ , \\ 0 & else\ , \end{cases}$$

$$\beta_{c,n}(y+y',y) := \begin{cases} \beta_c(y+y',y)\ , & (y,y') \in A_n \cap B_n\ , \\ 0 & else\ , \end{cases}$$

$$\beta_{s,n}(y+y',y) := \begin{cases} \beta_s(y+y',y)\ , & (y,y') \in C_n\ , \\ 0 & else\ , \end{cases}$$

$$K_n(y,y') := \begin{cases} K(y,y')\ , & (y,y') \in (A_n \cap B_n) \cup C_n\ , \\ 0 & else\ . \end{cases}$$

Then K_n is symmetric and γ_n, $\beta_{c,n}$, $\beta_{s,n}$ satisfy Hypothesis (H_{12}). Furthermore,

$$\gamma_n \nearrow \gamma\ , \quad \beta_{c,n} \nearrow \beta_c\ , \quad \beta_{s,n} \nearrow \beta_s\ , \quad K_n \nearrow K\ , \tag{3.26}$$

pointwise on the domains of γ, β_c, β_s, and K. Finally, the truncated kernels satisfy the detailed balance condition (H_{14}) with the same function H and the same probabilities P and Q.
Recall that V is given by (3.16) and define $\mathcal{J}$ by (3.17). For $v \in L_1^+$ put

$$D(v) := \frac{1}{2}\int_{\mathcal{E}} \mathcal{J}\Big(P(y,y')K(y,y')v(y)v(y')\ ,\ \gamma(y+y',y)v(y+y')\Big)\ d(y,y')\ ,$$

$$F(v) := \frac{1}{8}\int_{\mathcal{W}} \mathcal{J}\Big(\beta_c(y+y',y)Q(y'',y+y'-y'')K(y'',y+y'-y'')v(y'')v(y+y'-y'')\ ,$$
$$\beta_c(y+y',y'')Q(y,y')K(y,y')v(y)v(y')\Big)\ d(y,y',y'')\ ,$$

$$G(v) := \frac{1}{8}\int_{\mathcal{S}} \mathcal{J}\Big(\beta_s(y+y',y)K(y'',y+y'-y'')v(y'')v(y+y'-y'')\ ,$$
$$\beta_s(y+y',y'')K(y,y')v(y)v(y')\Big)\ d(y,y',y'')\ .$$

Finally, $D_n(v), F_n(v)$, and $G_n(v)$ are defined analogously but with $(\gamma_n, \beta_{c,n}, \beta_{s,n}, K_n)$ instead of $(\gamma, \beta_c, \beta_s, K)$.

Proposition 3.11. *Suppose that $u^0 \in L_1^+$ with $V(u^0) < \infty$ and denote by $u = u(\cdot; u^0)$ the unique solution of $(*)$ in $C^1(\mathbb{R}^+, L_1^+)$. Then, for any $t \geq s \geq 0$,*

$$0 \leq V\big(u(t)\big) \leq V\big(u(s)\big) < \infty \ . \tag{3.27}$$

Moreover, it holds

$$\big[\sigma \mapsto \varphi\big(u(\sigma)\big) D\big(u(\sigma)\big)\big] \in L_1(\mathbb{R}^+) \ . \tag{3.28}$$

Proof. For each $n \geq 1$ define

$$u_n^0(y) := \min\big\{n,\, \max\big\{u^0(y),\, H(y)/n\big\}\big\} \ , \quad y \in Y \ ,$$

so that by Hypothesis (H_{14})

$$0 < \frac{h_0}{n} \leq u_n^0 \leq n \quad \text{a.e.}$$

for any $n \geq 1$, since we may assume $h_0 \leq 1$. Then we claim that

$$V(u_n^0) \leq c(u^0) \ , \quad n \geq 1 \ , \tag{3.29}$$

for some constant $c(u^0) > 0$ being independent of $n \geq 1$. For, by definition of u_n^0 we have

$$\begin{aligned}
\int_0^{y_0} u_n^0 \log \frac{u_n^0}{H}\, dy &= \int_{\left[u^0 < \frac{H}{n} < n\right]} \frac{H}{n} \log \frac{1}{n}\, dy + \int_{\left[\frac{H}{n} \leq u^0 < n\right]} u^0 \log \frac{u^0}{H}\, dy \\
&\quad + \int_{\left[\frac{H}{n} < n \leq u^0\right]} n \log \frac{n}{H}\, dy + \int_{\left[n \leq \frac{H}{n}\right]} n \log \frac{n}{H}\, dy \\
&\leq \int_{\left[\frac{H}{n} \leq u^0 < n\right]} u^0 \log \frac{u^0}{H}\, dy + \int_{\left[\frac{H}{n} < n \leq u^0\, ,\, \frac{n}{H} > 1\right]} n \log \frac{n}{H}\, dy \\
&\leq \Big(\int_{S_n} + \int_{T_n} \Big) u^0 \log \frac{u^0}{H}\, dy \ ,
\end{aligned}$$

where we put

$$S_n := \Big[\frac{H}{n} \leq u^0 < n\Big] \ , \quad T_n := \big[n \leq u^0\big] \cap \Big[\frac{n}{H} > 1\Big] \ .$$

The last inequality is due to the fact that $x \mapsto x \log x$ is increasing on the interval $(1, \infty)$. Since (3.18) and $V(u^0) < \infty$ imply $u^0 \log \frac{u^0}{H} \in L_1$ and since $u_n^0 \to u^0$ in L_1, Lebesgue's theorem applies to give

$$\limsup_n V(u_n^0) \leq V(u^0) \tag{3.30}$$

from which (3.29) follows.
Next, consider for each $n \geq 1$ the problem

$$\dot{w} = \varphi(w) L_n[w] \ , \quad t > 0 \ , \qquad w(0) = u_n^0 \ , \tag{3.31}$$

where the operator L_n is defined by

$$L_n := L_{b,n} + L_{c,n} + L_{s,n} := \tilde{L}_b(\gamma_n) + \tilde{L}_c(\beta_{c,n}, K_n) + \tilde{L}_s(\beta_{s,n}, K_n)$$

as in (3.22). In virtue of Proposition 3.10, Problem (3.31) possesses a unique solution $u_n := u_n(\cdot; u_n^0) \in C^1(\mathbb{R}^+, L_\infty^+)$ such that for any $T > 0$ there exist constants $r_n^j(T)$ with

$$0 < r_n^1(T) \leq u_n(t) \leq r_n^2(T) < \infty \quad \text{a.e.} \ , \quad 0 \leq t \leq T \ . \tag{3.32}$$

Our next step is to prove that $V\big(u_n(\cdot)\big)$ decreases in time. Fix $T>0$. Due to (3.32) we may differentiate under the integral sign to obtain

$$\frac{d}{dt}V\big(u_n(t)\big) = \varphi\big(u_n(t)\big)\int_0^{y_0}\log\frac{u_n(t,y)}{H(y)}L_n[u_n(t)](y)\ dy \tag{3.33}$$

for $n\geq 1$ and $0\leq t\leq T$. We now have to compute the right hand-side of the above equality. It is an easy exercise to show that for $n\geq 1$ and $0\leq t\leq T$ it holds

$$\int_0^{y_0}\log\frac{u_n(t,y)}{H(y)}\big\{L_{b,n}[u_n(t)](y)+L_{c,n}^{(P)}[u_n(t)](y)\big\}\ dy = -D_n\big(u_n(t)\big)\ , \tag{3.34}$$

where $L_{c,n}^{(P)}$ consists of those integral terms of $L_{c,n}$ involving P but not Q. Note that Fubini's theorem applies throughout in the sequel because of (3.32). We then compute

$$\begin{aligned}
&\int_0^{y_0}\log\frac{u_n(t,y)}{H(y)}L_{s,n}[u_n(t)](y)\ dy\\
&\quad=\frac{1}{2}\int_{\mathcal{S}}\Big\{\log\frac{u_n(y'')}{H(y'')}-\log\frac{u_n(y)}{H(y)}\Big\}\beta_{s,n}(y+y',y'')K_n(y,y')u_n(y)u_n(y')\ d(y,y',y'')\\
&\quad=\frac{1}{4}\int_{\mathcal{S}}\Big\{\log\frac{u_n(y'')u_n(y+y'-y'')}{H(y'')H(y+y'-y'')}-\log\frac{u_n(y)u_n(y')}{H(y)H(y')}\Big\}\\
&\qquad\qquad\beta_{s,n}(y+y',y'')K_n(y,y')u_n(y)u_n(y')\ d(y,y',y'')\ ,
\end{aligned} \tag{3.35}$$

where we have taken into account $\beta_{s,n}(y,y')=\beta_{s,n}(y,y-y')$ and the symmetry of K_n. Next use the transformation $\mathcal{S}\to\mathcal{S}$, $(y,y',y'')\mapsto(y'',y+y'-y'',y)$ to deduce that the right hand side of (3.35) is equal to

$$\begin{aligned}
&\frac{1}{8}\int_{\mathcal{S}}\Big\{\log\frac{u_n(y'')u_n(y+y'-y'')}{H(y'')H(y+y'-y'')}-\log\frac{u_n(y)u_n(y')}{H(y)H(y')}\Big\}\\
&\qquad\qquad\beta_{s,n}(y+y',y'')K_n(y,y')u_n(y)u_n(y')\ d(y,y',y'')\\
&\quad+\frac{1}{8}\int_{\mathcal{S}}\Big\{\log\frac{u_n(y)u_n(y')}{H(y)H(y')}-\log\frac{u_n(y'')u_n(y+y'-y'')}{H(y'')H(y+y'-y'')}\Big\}\\
&\qquad\qquad\beta_{s,n}(y+y',y)K_n(y'',y+y'-y'')u_n(y'')u_n(y+y'-y'')\ d(y,y',y'')\ .
\end{aligned}$$

Finally, due to the detailed balance condition we may rewrite this last term to get

$$\int_0^{y_0}\log\frac{u_n(t,y)}{H(y)}L_{s,n}[u_n(t)](y)\ dy = -G_n\big(u_n(t)\big)\ . \tag{3.36}$$

Likewise one derives

$$\int_0^{y_0}\log\frac{u_n(t,y)}{H(y)}L_{c,n}^{(Q)}[u_n(t)](y)\ dy = -F_n\big(u_n(t)\big)\ , \tag{3.37}$$

where $L_{c,n}^{(Q)}$ are those integral terms of $L_{c,n}$ involving Q but not P. Putting this calculations together we deduce from (3.29) and (3.33)-(3.37)

$$V\big(u_n(t)\big)+\int_0^t\varphi\big(u_n(\sigma)\big)\big\{D_n\big(u_n(\sigma)\big)+F_n\big(u_n(\sigma)\big)+G_n\big(u_n(\sigma)\big)\big\}\ d\sigma = V(u_n^0)\leq c(u^0) \tag{3.38}$$

for all $n \geq 1$ and $0 \leq t \leq T$. Note then that for $0 \leq \sigma \leq T$ each of the terms $D_n\big(u_n(\sigma)\big), F_n\big(u_n(\sigma)\big)$, and $G_n\big(u_n(\sigma)\big)$ is non-negative. Thus we conclude

$$V\big(u_n(t)\big) \leq c(u^0) \ , \quad n \geq 1 \ , \quad t \geq 0 \ , \tag{3.39}$$

since $T > 0$ was arbitrary and $c(u^0)$ does not depend on T. But this bound and Lemma 3.8 together with the Dunford-Pettis theorem [**25**, Thm.4.21.2] imply that the set $\big\{u_n(t)\,;\, n \geq 1\big\}$ is relatively weakly compact in L_1 for each $t \geq 0$. In particular,

$$|u_n(t)|_1 \leq c(u^0) \ , \quad n \geq 1 \ , \quad t \geq 0 \ . \tag{3.40}$$

In the next step we prove that the set $\big\{u_n\,;\, n \geq 1\big\}$ is equicontinuous. Due to (3.26) and Hypotheses $(H_2) - (H_4)$ it follows as in Lemma 2.1 that there exists a constant c_0 being independent of $n \geq 1$ such that

$$|L_n[v]|_1 \leq c_0\big(1 + |v|_1\big)|v|_1 \ , \quad v \in L_1 \ , \quad n \geq 1 \ . \tag{3.41}$$

From the integral version of (3.31) and from (3.40) we infer then

$$|u_n(t) - u_n(s)|_1 \leq c(u^0)|t - s| \ , \quad t, s \geq 0 \ , \quad n \geq 1 \ . \tag{3.42}$$

In particular, the set $\big\{u_n\,;\, n \geq 1\big\}$ is equicontinuous with respect to the $L_{1,\mathrm{w}}$-topology. Now fix again $T > 0$ arbitrarily. Then the Arzelà-Ascoli theorem (see [**68**, Thm.1.3.2]) entails that there exist $\bar{u} \in C([0,T], L_{1,\mathrm{w}})$ and a subsequence (n_j) such that

$$u_j := u_{n_j} \rightarrow \bar{u} \quad \text{in} \quad C([0,T], L_{1,\mathrm{w}}) \ . \tag{3.43}$$

From (3.43) and the appendix (see Lemmas A.2 - A.4) it follows

$$L_j[u_j(\sigma)] \rightarrow L[\bar{u}(\sigma)] \quad \text{in} \quad L_{1,\mathrm{w}} \ , \quad 0 \leq \sigma \leq T \ . \tag{3.44}$$

Here, the operator $L = L_b + L_c + L_s$ is defined as in $(*)$. Inequality (3.42) and (3.43) give $\bar{u} \in C^{1-}([0,T], L_1)$. Since φ is weakly sequentially continuous, Lebesgue's theorem, (3.40), (3.41), and (3.44) yield as in the proof of Proposition 3.3 that

$$\int_0^t \varphi\big(u_j(\sigma)\big) L_j[u_j(\sigma)]\, d\sigma \longrightarrow \int_0^t \varphi\big(\bar{u}(\sigma)\big) L[\bar{u}(\sigma)]\, d\sigma \quad \text{in} \quad L_{1,\mathrm{w}} \ , \quad 0 \leq t \leq T \ .$$

Since $u_n^0 \rightarrow u^0$ in L_1, it thus follows from (3.43) that

$$\bar{u}(t) = u^0 + \int_0^t \varphi\big(\bar{u}(\sigma)\big) L[\bar{u}(\sigma)]\, d\sigma \ , \quad 0 \leq t \leq T \ .$$

Therefore, $\bar{u} \in C([0,T], L_1)$ is a mild solution of $(*)$. But this problem possesses a unique (mild) solution. Hence $\bar{u} = u(\cdot; u^0)\big|_{[0,T]}$ so that

$$u_j \rightarrow u(\cdot; u^0) \quad \text{in} \quad C([0,T], L_{1,\mathrm{w}}) \ . \tag{3.45}$$

Since V is weakly lower semi-continuous, we conclude from (3.30) and (3.38)

$$V\big(u(t)\big) \leq \liminf_j V\big(u_j(t)\big) \leq V(u^0) \ , \quad 0 \leq t \leq T \ .$$

The semiflow property implies then (3.27).
It remains to prove (3.28). It follows from (3.45) that

$$\gamma_j(y + y', y) u_j(\sigma, y + y') \rightarrow \gamma(y + y', y) u(\sigma, y + y') \ \text{ in } \ L_{1,\mathrm{w}}(\mathcal{E}) \ ,$$
$$P(y, y') K_j(y, y') u_j(\sigma, y) u_j(\sigma, y') \rightarrow P(y, y') K(y, y') u(\sigma, y) u(\sigma, y') \ \text{ in } \ L_{1,\mathrm{w}}(\mathcal{E}) \ ,$$

for $0 \le \sigma \le T$. We postpone the proof of this claim to the appendix (see Lemmas A.2 and A.3). From Lemma 3.7 we then obtain

$$0 \le D\big(u(t)\big) \le \liminf_j D_j\big(u_j(t)\big) \ , \quad 0 \le t \le T \ .$$

Since φ is weakly sequentially continuous, by Fatou's lemma and (3.38) we have

$$0 \le \int_0^t \varphi\big(u(\sigma)\big)D\big(u(\sigma)\big)\, d\sigma \le \liminf_j \int_0^t \varphi\big(u_j(\sigma)\big)D_j\big(u_j(\sigma)\big)\, d\sigma \le c(u^0) \ ,$$

for $0 \le t \le T$, whereby $c(u^0)$ does not depend on $T > 0$. □

Recall that the equilibria u_α, $\alpha \in \mathbb{R}$, are given by $u_\alpha(y) := H(y)e^{\alpha y}$, $y \in Y$. Clearly, for any given $\varrho > 0$, there exists exactly one $\alpha := \alpha(\varrho) \in \mathbb{R}$ such that $M(u_\alpha) = \varrho$, where the mass $M(v)$ of $v \in L_1^+$ is defined as

$$M(v) := \int_0^{y_0} yv(y)\, dy \ .$$

We can state now the main result concerning convergence towards equilibrium.

Theorem 3.12. *Let Hypotheses $(H_1) - (H_5)$ and $(H_{11}) - (H_{14})$ be satisfied. Suppose that $u^0 \in L_1^+ \setminus \{0\}$ with $V(u^0) < \infty$. For $\varrho := M(u^0)$ choose $\alpha := \alpha(\varrho) \in \mathbb{R}$ such that $M(u_\alpha) = \varrho$. Then, given any sequence $t_n \nearrow \infty$ and any $T > 0$, the solution $u = u(\cdot; u^0)$ of Problem $(*)$ satisfies*

$$u(\cdot + t_n; u^0) \to u_\alpha \quad in \quad C([0,T], L_{1,\mathrm{w}}) \ . \tag{3.46}$$

In particular, $u(t_n; u^0) \to u_\alpha$ in $L_{1,\mathrm{w}}$.
In addition, if there exists $r \in L_1^+$ such that for a.a. $y \in Y$

$$\gamma(\cdot, y) \le r(y) \quad a.e. \ on \quad (y, y_0) \tag{3.47}$$

and if $u^0 > 0$ a.e., then

$$u(\cdot + t_n; u^0) \to u_\alpha \quad in \quad C\big([0,T], L_1\big) \ . \tag{3.48}$$

In particular, $u(t_n; u^0) \to u_\alpha$ in L_1.

Proof. By the semiflow property we have

$$u_n(t) := u(t + t_n; u^0) = u\big(t; u(t_n; u^0)\big) \ , \quad t \ge 0 \ ,$$

so that $u_n \in C^1(\mathbb{R}^+, L_1^+)$. From Proposition 3.11 we know

$$V\big(u_n(t)\big) \le V(u^0) \ , \quad t \ge 0 \ , \quad n \ge 1 \ . \tag{3.49}$$

Analogously to the proof of Proposition 3.11 we deduce the existence of a function $\bar u \in C^{1-}([0,T], L_1)$ and a subsequence (n') such that $u_{n'} \to \bar u$ in $C([0,T], L_{1,\mathrm{w}})$. Obviously, $\bar u(t) \in L_1^+$ and $M\big(\bar u(t)\big) = \varrho$ for $0 \le t \le T$. As in the proof of Proposition 3.11 we infer

$$\begin{aligned} 0 &\le \int_0^T \varphi\big(\bar u(t)\big)D\big(\bar u(t)\big)\, dt \le \liminf_{n'} \int_0^T \varphi\big(u_{n'}(t)\big)D\big(u_{n'}(t)\big)\, dt \\ &\le \limsup_{n'} \int_{t_{n'}}^\infty \varphi\big(u(t)\big)D\big(u(t)\big)\, dt = 0 \ , \end{aligned} \tag{3.50}$$

since $\varphi(u)D(u) \in L_1(\mathbb{R}^+)$. In view of Hypothesis (H_{11}) we derive $D\big(\bar u(t)\big) = 0$ for a.a. $0 \le t \le T$. But this implies that for a.a. $0 \le t \le T$ the function $\bar u(t)$ satisfies the assumption of Lemma 3.9. Hence, for a.a. $0 \le t \le T$ there exists $\bar{\alpha}(t) \in \mathbb{R}$ such that $\bar{u}(t) = u_{\bar{\alpha}(t)}$ in L_1. Recalling $M\big(\bar{u}(t)\big) = \varrho$, $0 \le t \le T$, it follows by definition of $\alpha = \alpha(\varrho)$ that $\bar{\alpha}(t) = \alpha$ for all $0 \le t \le T$ since $\bar{u}$ is continuous. Consequently we have $u_{n'} \to u_\alpha$ in $C([0,T], L_{1,\mathrm{w}})$. Since this limit is independent of the subsequence (n'), we deduce statement (3.46).

Lastly, we have to show (3.48). Using (3.47) and (3.46) we see

$$L_b^1[u_n(t)](y) \to L_b^1[u_\alpha](y) \ , \quad \text{a.a. } y \in Y \ , \quad 0 \le t \le T \ .$$

Moreover, (3.49) and Lemma 3.8 lead to

$$|L_b^1[u_n(t)](y)| \le |u_n(t)|_1\, r(y) \le c_0\, r(y) \ , \quad \text{a.a. } y \in Y \ , \quad 0 \le t \le T \ ,$$

where $c_0 > 0$ depends neither on $n \in \mathbb{N}$ nor on $0 \le t \le T$. Lebesgue's theorem and (3.46) imply then

$$\varphi(u_n) L_b^1[u_n] \to \varphi(u_\alpha) L_b^1[u_\alpha] \quad \text{in} \quad L_1\big((0,T) \times Y\big) \ , \tag{3.51}$$

since φ is weakly sequentially continuous and bounded. Define for $v \in L_1$

$$h(v)(y) := \int_0^{y_0 - y} P(y,y')K(y,y')v(y')\, dy' \ , \quad y \in Y \ . \tag{3.52}$$

Analogously as above we have $h\big(u_n(t)\big)(y) \to h(u_\alpha)(y)$ and $|h\big(u_n(t)\big)(y)| \le \|K\|_\infty c_0$ for all $0 \le t \le T$ and a.a. $y \in Y$. Hence, again by Lebesgue's theorem,

$$\varphi(u_n) h(u_n) \to \varphi(u_\alpha) h(u_\alpha) \quad \text{in} \quad L_1\big((0,T) \times Y\big) \ . \tag{3.53}$$

Since $u^0 > 0$ a.e., Corollary 2.5 yields $u(t;u^0) > 0$ a.e. for $t \ge 0$. Fix $\lambda > 1$ and observe that for $\xi, \eta > 0$ the inequality

$$|\eta - \xi| \le (\lambda - 1)\xi + \frac{1}{\log \lambda}(\eta - \xi)(\log \eta - \log \xi)$$

holds, from which we derive

$$\begin{aligned}
&\big|\varphi(u_n) u_n h(u_n) - \varphi(u_n) L_b^1[u_n]\big|_{L_1((0,T)\times Y)} \\
&\le \int_0^T \varphi(u_n) \int_0^{y_0} \int_0^{y_0 - y} \big|P(y,y')K(y,y')u_n(y)u_n(y') - \gamma(y+y',y)u_n(y+y')\big|\, dy'dydt \\
&\le (\lambda - 1)\big|\varphi(u_n) L_b^1[u_n]\big|_{L_1((0,T)\times Y)} + \frac{2}{\log \lambda} \int_0^T \varphi(u_n) D(u_n)\, dt \ .
\end{aligned}$$

As in (3.50) the last integral of the above inequality converges towards zero as $n \nearrow \infty$. Taking the $\limsup_n$ on both sides and letting then λ tend to 1, we obtain from (3.51)

$$\varphi(u_n) u_n h(u_n) \to \varphi(u_\alpha) L_b^1[u_\alpha] = \varphi(u_\alpha) u_\alpha h(u_\alpha) \quad \text{in} \quad L_1\big((0,T) \times Y\big)$$

as $n \nearrow \infty$, whereby the equality is implied by the detailed balance condition. Therefore, we may extract a subsequence (n') of (n) such that $\varphi(u_{n'}) u_{n'} h(u_{n'}) \to \varphi(u_\alpha) u_\alpha h(u_\alpha)$ a.e. on $(0,T) \times Y$, and, in virtue of (3.53), we may assume that also $\varphi(u_{n'}) h(u_{n'}) \to \varphi(u_\alpha) h(u_\alpha)$ a.e. on $(0,T) \times Y$. Due to Hypotheses (H_{11}) and (H_{13}) we have $\varphi(u_\alpha) h(u_\alpha) > 0$ a.e. and thus $u_{n'} \to u_\alpha$ a.e. on $(0,T) \times Y$. Since (3.40) is valid here too, (3.46) implies $u_{n'} \to u_\alpha$ in $L_{1,\mathrm{w}}\big((0,T) \times Y\big)$ and a.e. on $(0,T) \times Y$. Hence, according to Vitali's theorem,

$$u_{n'} \to u_\alpha \quad \text{in} \quad L_1\big((0,T) \times Y\big) \ . \tag{3.54}$$

Together with Lemma 2.1 we conclude

$$L[u_{n'}] \to L[u_\alpha] = 0 \quad \text{in} \quad L_1\big((0,T) \times Y\big) \ . \tag{3.55}$$

Observing

$$u_{n'}(t) = u_{n'}(s) + \int_s^t \varphi\big(u_{n'}(\sigma)\big) L[u_{n'}(\sigma)]\, d\sigma\ , \quad 0 \le s \le t\ ,$$

we see that for each $t \in (0,T]$

$$\begin{aligned} t|u_{n'}(t) - u_\alpha|_1 &\le |u_{n'} - u_\alpha|_{L_1((0,T)\times Y)} + \varphi^* \int_0^t \int_s^t |L[u_{n'}(\sigma)]|_1\, d\sigma ds \\ &\le |u_{n'} - u_\alpha|_{L_1((0,T)\times Y)} + T\varphi^*\, |L[u_{n'}]|_{L_1((0,T)\times Y)}\ . \end{aligned}$$

In (3.54) and (3.55) we proved that the right hand side of the above estimate tends to zero as $n' \nearrow \infty$. Therefore, $u_{n'} \to u_\alpha$ in $C\big((0,T], L_1\big)$. This limit being independent of the subsequence (n'), assertion (3.48) is now evident. □

Remarks 3.13. (a) Theorem 3.12 implies that there exist no further equilibria in L_1^+ with finite 'entropy' V.
(b) Note that by

$$x|\log x| \le c(\varepsilon)(x^{1+\varepsilon} + x^{1-\varepsilon})\ , \quad x > 0\ ,$$

for fixed $\varepsilon > 0$, Hölder's inequality and Hypothesis (H_{14}) imply that $V(w) < \infty$ provided $w \in L_p^+$ with $p > 1$.
(c) Theorem 3.12 can be useful in applications. For instance, one may determine the kernels by observing the asymptotic distribution. Note that this asymptotic distribution depends only on the mass of the initial distribution but not on its shape which seems to be consistent with numerical simulations. We refer to [**27**], [**36**], [**42**], and [**66**] for details.

The purpose of the end of this part is to prove stability of the equilibria in an appropriate topology. To shorten notation, the following definitions are made. For $\varrho > 0$ we put

$$L_{1,\varrho}^+ := \big\{w \in L_1^+\,;\, M(w) = \varrho\big\}\ , \quad X^+ := \big\{u \in L_1^+\,;\, V(u) < \infty\big\}\ , \quad X_\varrho^+ := L_{1,\varrho}^+ \cap X^+\ .$$

If not stated otherwise, the topology of X^+ and X_ϱ^+ is the one induced from L_1. Thus they are metric spaces. The previous considerations show that X_ϱ^+ is positively invariant, that is, $u(t;u^0) \in X_\varrho^+$ for $t \ge 0$ provided $u^0 \in X_\varrho^+$. Observe that $L_p^+ \subset X^+$ for $p > 1$ due to Remarks 3.13(b). Finally, for any $\lambda \in \mathbb{R}$ set

$$V^\lambda(w) := V(w) - \int_0^{y_0} H(y)\, dy - \lambda M(w)\ , \quad w \in X^+\ .$$

Proposition 3.14. *For $\varrho > 0$ choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then, $u_{\alpha(\varrho)}$ is the unique minimizer of V on X_ϱ^+ and of $V^{\alpha(\varrho)}$ on X^+.*
Moreover, for any minimizing sequence (w_j) of V on X_ϱ^+, it holds $w_j \to u_{\alpha(\varrho)}$ in X_ϱ^+.

Proof. For $r > 0$ define

$$f_r(w) := w\big(\log\frac{w}{r} - 1\big)\ , \quad w \ge 0\ ,$$

with $f_r(0) := 0$. Then f_r has in $w = r$ a global minimum for each $r > 0$. For brevity put $\alpha := \alpha(\varrho)$. Given $w \in X^+$

$$V^\alpha(w) = \int_0^{y_0} f_{u_\alpha(y)}\big(w(y)\big)\, dy \ge \int_0^{y_0} f_{u_\alpha(y)}\big(u_\alpha(y)\big)\, dy = V^\alpha(u_\alpha)\ ,$$

where the inequality is strict provided $w \neq u_\alpha$. Hence, u_α is the unique minimizer of V^α on X^+. Furthermore, since $M(w) = \varrho$ for all $w \in X_\varrho^+$, it also minimizes V on X_ϱ^+.
Let now (w_j) be a minimizing sequence in X_ϱ^+ of V, i.e.

$$\lim V(w_j) = \inf_{w \in X_\varrho^+} V(w) = V(u_\alpha) \ . \tag{3.56}$$

Then $M(w_j) = \varrho = M(u_\alpha)$ for $j \in \mathbb{N}$ and thus $\lim V^\alpha(w_j) = V^\alpha(u_\alpha)$. Observing that

$$\left| f_{u_\alpha(\cdot)}\big(w_j(\cdot)\big) - f_{u_\alpha(\cdot)}\big(u_\alpha(\cdot)\big)\right|_1 = \ V^\alpha(w_j) - V^\alpha(u_\alpha) \longrightarrow 0$$

as $j \to \infty$, we may extract a subsequence (j') such that $f_{u_\alpha(\cdot)}\big(w_{j'}(\cdot)\big) \to f_{u_\alpha(\cdot)}\big(u_\alpha(\cdot)\big)$ a.e.. This easily implies $w_{j'} \to u_\alpha$ a.e.. Owing to (3.56) there exists $c > 0$ with $V(w_{j'}) \leq c$ for each j'. Thus, from Lemma 3.8 and the Dunford-Pettis theorem we deduce that $(w_{j'})$ is relatively weakly compact in L_1. Therefore, there exists a further subsequence (j'') and $w \in L_1$ such that $w_{j''} \to w$ in $L_{1,\mathrm{w}}$. Since $w_{j''} \in L_{1,\varrho}^+$, we have $w \in L_{1,\varrho}^+$. V being weakly lower semi-continuous,

$$V(w) \leq \liminf_{j''} V(w_{j''}) = V(u_\alpha) < \infty \ .$$

By what was proved above, $w = u_\alpha$. Altogether, we obtain $w_{j''} \to u_\alpha$ in $L_{1,\mathrm{w}}$ and a.e. so that, due to Vitali's theorem, $w_{j''} \to u_\alpha$ in L_1^+. This limit being independent of the subsequences, the assertion follows. □

Theorem 3.15. *Let $\varrho > 0$ be given and choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u^0 \in X_\varrho^+$ with $|u^0 - u_{\alpha(\varrho)}|_1 < \delta$ and $V(u^0) < V(u_{\alpha(\varrho)}) + \delta$ it holds $|u(t;u^0) - u_{\alpha(\varrho)}|_1 < \varepsilon$ for $t \geq 0$.*

Proof. Due to [**17**, Prop.4.3] we only have to show that V is decreasing along orbits - which was done in Proposition 3.11 - and that $u_{\alpha(\varrho)}$ lies in a 'potential well' with respect to X_ϱ^+, that is, for given small $\varepsilon > 0$ there exists $\sigma(\varepsilon) > 0$ such that $V(w) - V(u_{\alpha(\varrho)}) \geq \sigma(\varepsilon)$ for all $w \in X_\varrho^+$ with $|w - u_{\alpha(\varrho)}|_1 = \varepsilon$. But this easily follows from Proposition 3.14. □

Define

$$d(w,v) := |w - v|_1 + |V(w) - V(v)| \ , \quad w,v \in X^+ \ .$$

Then, for any $\varrho > 0$, (X_ϱ^+, d) is a metric space with a stronger topology than the L_1-topology.

Corollary 3.16. *Let $\varrho > 0$ be arbitrary and choose $\alpha(\varrho) \in \mathbb{R}$ such that $M(u_{\alpha(\varrho)}) = \varrho$. Then the equilibrium $u_{\alpha(\varrho)}$ is stable in (X_ϱ^+, d), that is, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u^0 \in X_\varrho^+$ with $d(u^0, u_{\alpha(\varrho)}) < \delta$ it holds $d\big(u(t;u^0), u_{\alpha(\varrho)}\big) < \varepsilon$ for $t \geq 0$.*

Proof. Since V decreases along orbits, this follows from the above theorem. □

At this point it may be worthwhile to give some examples of kernels satisfying the imposed assumptions.

Examples 3.17. (a) If φ is defined as in Remark 3.5, then it satisfies Hypotheses (H_1) and (H_{11}) .

(b) Let $P \in C\big(Y \times Y, (0,\infty)\big)$ be symmetric and let $q \in C(Y, \mathbb{R}^+)$ be such that

$$0 < P(y,y') + q(y+y') \leq 1 \ , \quad 0 < y + y' \leq y_0 \ .$$

Assume that $\alpha \geq 0$ and $0 \geq \alpha - \beta > -1$ and define for arbitrary constants $K^*, \gamma^* > 0$

$$Q(y,y') := q(y+y')\ , \quad 0 < y+y' \leq y_0\ ,$$
$$K(y,y') := K^*(y+y')^\alpha\ , \quad 0 < y, y' \leq y_0\ ,$$
$$\gamma(y,y') := \gamma^* P(y-y',y') y^\beta \big[y'(y-y')\big]^{\alpha-\beta}\ , \quad 0 < y' < y \leq y_0\ ,$$
$$\beta_c(y,y') := \big(\mathsf{B}(\alpha-\beta+2, \alpha-\beta+1)\big)^{-1} y^{-1-2\alpha+2\beta} \big[y'(y-y')\big]^{\alpha-\beta}\ , \quad 0 < y' < y \leq y_0\ ,$$
$$\beta_s(y,y') := f_s(y) \big[y'(y-y')\big]^{\alpha-\beta}\ , \quad 0 < y - y_0 < y' \leq y_0\ ,$$

with B denoting the beta function and

$$f_s(y) := y\Big(\int_{y-y_0}^{y_0} y' \big[y'(y-y')\big]^{\alpha-\beta}\ dy'\Big)^{-1}\ , \quad y_0 < y < 2y_0\ .$$

Then Hypotheses $(H_2) - (H_5)$ and $(H_{12}) - (H_{14})$ are satisfied with

$$H(y) := \frac{\gamma^*}{K^*} y^{\alpha-\beta}\ , \quad y \in Y\ .$$

Further, (3.47) holds provided $\alpha = \beta$. In addition, if also $P \equiv \mathit{const}$ then $\gamma(y,y'), \beta_c(y,y')$, and $\beta_s(y,y')$ are independent of y' (compare this result with the 'power-law breakup' of Examples 2.21).

(c) Analogously as in [**39**] we may define

$$\begin{aligned} K(y,y') &:= re^{-y^2-(y')^2}\ , && 0 < y, y' \leq y_0\ ,\\ \gamma(y,y') &:= se^{-(y-2y')^2}\ , && 0 < y' < y \leq y_0\ ,\\ \beta_s(y,y') &:= f(y) e^{-4y(y-y')}\ , && 0 < y - y_0 < y' \leq y_0\ , \end{aligned}$$

for some $r, s > 0$, where

$$f(y) := y\Big(\int_{y-y_0}^{y_0} y'' e^{-4y(y-y'')}\ dy''\Big)^{-1}\ , \quad y_0 < y < 2y_0\ .$$

Then, for $P \equiv 1$ and $Q \equiv 0$, Hypotheses $(H_2) - (H_5)$ and $(H_{12}) - (H_{14})$ hold with

$$H(y) := \frac{s}{r} e^{-y^2}\ , \quad y \in Y\ ,$$

and, in addition, (3.47) is satisfied.

(d) The other example from [**39**] can also be considered. Let α, τ, p, λ be arbitrary real numbers and let $A_0, B_0 > 0$. Put

$$K(y,y') := A_0(1+y)^\alpha(1+y')^\alpha\ ,$$
$$\gamma(y,y') := B_0 K(y', y-y')(1+y)^\tau \big[(1+y')(1+y-y')\big]^{-\tau} e^{\lambda(y^p-(y-y')^p-(y')^p)}\ ,$$
$$\beta_s(y,y') := y\nu(y,y')\Big(\int_{y-y_0}^{y_0} y''\nu(y,y'')\ dy''\Big)^{-1}\ ,$$

where $\nu(y,z) := (1+z)^{\alpha-\tau}(1+y-z)^{\alpha-\tau} e^{-\lambda(z^p+(y-z)^p)}$. Then, with $P \equiv 1$, $Q \equiv 0$, and

$$H(y) := \frac{B_0}{A_0}(1+y)^{-\tau} e^{-\lambda y^p - y}\ , \quad y \in Y\ ,$$

Hypotheses $(H_2) - (H_5)$ and $(H_{12}) - (H_{14})$ inclusive (3.47) are satisfied.

Appendix

We give in this appendix the proofs of some technical weak convergence results used in the proof of Proposition 3.3 and Proposition 3.11.

Recall that the sets Δ and Λ have been defined in Hypotheses (H_2) and (H_4), respectively, and that $\mathcal{E} := \big\{(y,y') \in Y^2\,;\, 0 < y + y' < y_0\big\}$. Assume that the kernels $\gamma, \beta_c, \beta_s, K, P$, and Q satisfy Hypotheses $(H_2) - (H_5)$ and (H_{12}) and that

$$\gamma_n \nearrow \gamma\ , \quad \beta_{c,n} \nearrow \beta_c \quad \text{on} \quad \Delta\ , \tag{A.1}$$

$$\beta_{s,n} \nearrow \beta_s \quad \text{on} \quad \Lambda\ , \qquad K_n \nearrow K \quad \text{on} \quad Y \times Y \tag{A.2}$$

are valid as $n \to \infty$. Furthermore, let

$$w_n \to w \quad \text{in} \quad L_{1,\mathrm{w}}\ , \tag{A.3}$$

with $|w_n|_1 \le c_0$.

The first proposition is implicitly contained in [**57**, Lem.4.1]. Anyhow, for the reader's ease, we give here its proof.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^m$, $m \ge 1$, be a measurable set with measure $|\Omega| > 0$. Assume that $h_n, h \in L_\infty(\Omega)$ are such that $\|h_n\|_\infty \le c_1$ for $n \ge 1$ and $h_n \to h$ a.e.. Then, provided $v_n \to v$ in $L_{1,\mathrm{w}}(\Omega)$, it holds $h_n v_n \to hv$ in $L_{1,\mathrm{w}}(\Omega)$.*

Proof. Let $\varepsilon > 0$ and $f \in L_\infty(\Omega), f \ne 0$, be arbitrary. According to the Dunford-Pettis theorem we find $\delta := \delta(\varepsilon) > 0$ such that for each measurable subset A of Ω with measure $|A| \le \delta$ we have

$$\int_A |v_n(x)|\ dx \le \frac{\varepsilon}{(c_1 + \|h\|_\infty)\|f\|_\infty}\ , \quad n \ge 1\ .$$

Further, since h_n converges towards h a.e., by Egoroff's theorem we may choose $A \subset \Omega$ with measure $|A| \le \delta$ such that $h_n \to h$ uniformly on $\Omega \setminus A$ is valid and hence (see [**15**, p.89]) $h_n \to h$ in $L_\infty(\Omega \setminus A)$. Therefore, we deduce

$$\begin{aligned}
&\Big|\int_\Omega f(x)\big[h_n(x)v_n(x) - h(x)v(x)\big]\ dx\Big| \\
&\quad\le \Big|\int_A f(x)v_n(x)\big[h_n(x) - h(x)\big]\ dx\Big| + \Big|\int_{\Omega\setminus A} f(x)v_n(x)\big[h_n(x) - h(x)\big]\ dx\Big| \\
&\qquad + \Big|\int_\Omega f(x)h(x)\big[v_n(x) - v(x)\big]\ dx\Big| \\
&\quad\le \|f\|_\infty(c_1 + \|h\|_\infty)\int_A |v_n(x)|\ dx\ + \|f\|_\infty \sup_k \|v_k\|_{L_1(\Omega)}\, \|h_n - h\|_{L_\infty(\Omega\setminus A)} \\
&\qquad + \Big|\int_\Omega f(x)h(x)\big[v_n(x) - v(x)\big]\ dx\Big|\ .
\end{aligned}$$

Letting $n \to \infty$ one gets

$$0 \le \limsup_n \Big|\int_\Omega f(x)\big[h_n(x)v_n(x) - h(x)v(x)\big]\ dx\Big| \le \varepsilon$$

and hence the assertion.

$\square$

Recall that the operator $L_b := L_b(\tilde{\gamma})$ is defined by

$$L_b(\tilde{\gamma})[v](y) := \int_y^{y_0} \tilde{\gamma}(y',y)v(y')\,dy' - \frac{1}{2}v(y)\int_0^y \tilde{\gamma}(y,y')\,dy' .$$

Lemma A.2. *(i) Defining for $(y,y') \in \mathcal{E}$*

$$\begin{aligned} v_n(y,y') &:= \gamma_n(y+y',y)w_n(y+y') , \\ v(y,y') &:= \gamma(y+y',y)w(y+y') , \end{aligned}$$

it holds $v_n \to v$ in $L_{1,\mathrm{w}}(\mathcal{E})$.
(ii) The convergence $L_b(\gamma_n)[w_n] \to L_b(\gamma)[w]$ in $L_{1,\mathrm{w}}$ is valid.

Proof. (i) Using the transformation $(y,y') \mapsto (y', y-y')$, Fubini's theorem, and the triangle inequality, we see that for $f \in L_\infty(\mathcal{E})$

$$\begin{aligned} &\Big|\int_{\mathcal{E}} f(y,y')\big[v_n(y,y') - v(y,y')\big]\,d(y,y')\Big| \\ &\qquad \le \|f\|_\infty \int_0^{y_0} a_n(y)|w(y)|\,dy + \Big|\int_0^{y_0} h_n(y)\big[w(y)-w_n(y)\big]\,dy\Big| \end{aligned} \tag{A.4}$$

holds, where

$$a_n(y) := \int_0^y |\gamma_n(y,y') - \gamma(y,y')|\,dy' , \qquad h_n(y) := \int_0^y f(y', y-y')\gamma_n(y,y')\,dy' .$$

Due to Hypothesis (H_2) and (A.1), an application of Lebesgue's theorem yields that the first term on the right hand side of (A.4) converges to 0 as $n \to \infty$. Next observe that for a.a. $y \in Y$ we have $f(\cdot, y-\cdot) \in L_\infty\big((0,y)\big)$ with $\|f(\cdot,y-\cdot)\|_{L_\infty((0,y))} \le \|f\|_\infty$, due to Fubini's theorem. We obtain $\|h_n\|_{L_\infty(Y)} \le \|f\|_\infty m_\gamma$ and, by Lebesgue's theorem,

$$h_n(y) \to h(y) := \int_0^y f(y', y-y')\gamma(y,y')\,dy' , \quad \text{a.a. } y \in Y ,$$

where $h \in L_\infty(Y)$. Proposition A.1 now yields (i).
(ii) Using Lemma 2.7, this follows similarly to (i). □

Next we consider the collision operator L_c. Recall that

$$\begin{aligned} \tilde{L}_c(\beta_{c,n},K_n)[v](y) &:= \tilde{L}_c^1(K_n)[v](y) + \tilde{L}_c^2(\beta_{c,n},K_n)[v](y) - \tilde{L}_c^3(\beta_{c,n},K_n)[v](y) \\ &:= \frac{1}{2}\int_0^y P(y',y-y')K_n(y',y-y')v(y')v(y-y')\,dy' \\ &\quad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} Q(y'',y'-y'')K_n(y'',y'-y'')\beta_{c,n}(y',y)v(y'')v(y'-y'')\,dy''dy' \\ &\quad - v(y)\int_0^{y_0-y}\Big\{P(y,y') + \frac{1}{2}\int_0^{y+y'}\beta_{c,n}(y+y',y'')\,dy''\,Q(y,y')\Big\}K_n(y,y')v(y')\,dy' , \end{aligned}$$

and that

$$\begin{aligned}L_c[v](y) &:= L_c^1[v](y) + L_c^2[v](y) - L_c^3[v](y)\\ &:= \frac{1}{2}\int_0^y P(y',y-y')K(y',y-y')v(y')v(y-y')\ dy'\\ &\quad + \frac{1}{2}\int_y^{y_0}\int_0^{y'} Q(y'',y'-y'')K(y'',y'-y'')\beta_c(y',y)v(y'')v(y'-y'')\ dy''dy'\\ &\quad - v(y)\int_0^{y_0-y} K(y,y')\big\{P(y,y')+Q(y,y')\big\}v(y')\ dy'\ .\end{aligned}$$

Lemma A.3. *(i) Defining for $(y,y')\in\mathcal{E}$*

$$\begin{aligned}v_n(y,y') &:= P(y,y')K_n(y,y')w_n(y)w_n(y')\ ,\\ v(y,y') &:= P(y,y')K(y,y')w(y)w(y')\ ,\end{aligned}$$

it holds $v_n\to v$ in $L_{1,\mathrm{w}}(\mathcal{E})$.
(ii) The convergence $\tilde{L}_c(\beta_{c,n},K_n)[w_n]\to L_c[w]$ in $L_{1,\mathrm{w}}$ is valid.

Proof. Without restricting generality we may assume that $P\equiv 1$ and $Q\equiv 1$.
(i) Let $f\in L_\infty(\mathcal{E})$ be arbitrary. One easily deduces

$$\begin{aligned}&\Big|\int_{\mathcal{E}} f(y,y')\big[v_n(y,y')-v(y,y')\big]\ d(y,y')\Big|\\ &\qquad\le \Big|\int_0^{y_0} h_n(y)\big[w_n(y)-w(y)\big]\ dy\Big| + \int_0^{y_0}|g_n(y)|\ |w(y)|\ dy\\ &\qquad\quad + \|f\|_\infty\int_{\mathcal{E}}|K_n(y,y')-K(y,y')|\ |w(y')|\ |w(y)|\ d(y,y')\\ &\qquad =: I_n + II_n + III_n\ ,\end{aligned}$$

where

$$h_n(y) := \int_0^{y_0-y} f(y,y')K_n(y,y')w_n(y')\ dy'\ ,$$

and

$$g_n(y) := \int_0^{y_0-y} f(y,y')K_n(y,y')\big[w_n(y')-w(y')\big]\ dy'\ .$$

For a.a. $y\in Y$ we have

$$\|f(y,\cdot)K_n(y,\cdot)\|_{L_\infty((0,y_0-y))}\le\|f\|_\infty\,\|K\|_\infty\ ,\quad n\ge 1\ ,$$

so that Proposition A.1 and (A.2) imply $h_n\to h$ a.e. on Y where h is defined as h_n but with K and w instead of K_n and w_n. Further, $h_n, h\in L_\infty(Y)$ with $\|h_n\|_\infty\le\|f\|_\infty\|K\|_\infty c_0$ for $n\ge 1$. Applying Proposition A.1 once more we obtain $I_n\to 0$ as $n\to\infty$. Since analogously $g_n\to 0$ a.e. on Y and

$$|g_n(y)|\le\|f\|_\infty\,\|K\|_\infty(c_0+|w|_1)\ ,\quad \text{a.a. } y\in Y\ ,$$

Lebesgue's theorem yields $II_n\to 0$ as $n\to\infty$. Finally, $III_n\to 0$ also follows from the latter.
(ii) For the remainder of this proof fix $f\in L_\infty(Y)$ arbitrarily. Using Lemma 2.7 one shows analogously to (i) that $\tilde{L}_c^1(K_n)[w_n]\to L_c^1[w]$ in $L_{1,\mathrm{w}}$. For the sake of completeness we prove

$$\tilde{L}_c^2(\beta_{c,n},K_n)[w_n]\to L_c^2[w]\quad\text{in}\quad L_{1,\mathrm{w}} \tag{A.5}$$

in detail. With the aid of the transformation $(y,y',y'') \mapsto (y'', y+y'-y'', y)$ and Fubini's theorem we infer

$$\begin{aligned}
\Big|\int_0^{y_0} f(y)\big\{\tilde{L}_c^2(\beta_{c,n},K_n)[w_n](y) - L_c^2[w](y)\big\}\ dy\Big| \\
&\le \Big|\int_0^{y_0} h_n(y)\big[w_n(y)-w(y)\big]\ dy\Big| + \int_0^{y_0} |g_n(y)|\ |w(y)|\ dy \\
&\quad + \|f\|_\infty \int_0^{y_0}\int_0^{y_0-y} b_n(y,y')|w(y')|\ dy'\ |w(y)|\ dy \\
&=: I_n + II_n + III_n
\end{aligned}$$

where

$$h_n(y) := \int_0^{y_0-y}\int_0^{y+y'} f(y'')\beta_{c,n}(y+y',y'')\ dy''\ K_n(y,y')w_n(y')\ dy'$$

and

$$g_n(y) := \int_0^{y_0-y}\int_0^{y+y'} f(y'')\beta_{c,n}(y+y',y'')\ dy''\ K_n(y,y')\big[w_n(y')-w(y')\big]\ dy'$$

as well as

$$b_n(y,y') := \int_0^{y+y'} \big|\beta_{c,n}(y+y',y'')K_n(y,y') - \beta_c(y+y',y'')K(y,y')\big|\ dy''\ .$$

We now show $I_n \to 0$ as $n \to \infty$. Putting $K_0 := K$ we can find according to (A.2) a subset A of Y such that $|A^c| = 0$ and $\|K_n(y,\cdot)\|_{L_\infty(Y)} \le \|K\|_\infty$ for $n \in \mathbb{N}$ and $y \in A$. Observe then that due to Hypothesis (H_3), (3.14), and [**13**, X.Thm.6.7, X.Lem.7.2] the map

$$\mathcal{E} \to \mathbb{R}^+\ , \quad (y,y') \mapsto \int_0^{y+y'} \beta_c(y+y',y'')\ dy''$$

is measurable and equals 2 a.e.. Hence, [**13**, X.Kor.6.8] entails that there exists $N \subset Y$ with $|N^c| = 0$ and the property that for each $y \in N$ there exists a set $B_y \subset (0, y_0 - y)$ such that $|B_y^c| = 0$ and

$$\int_0^{y+y'} \beta_c(y+y',y'')\ dy'' = 2\ , \quad y' \in B_y\ . \tag{A.6}$$

Put $Z := A \cap N \subset Y$ and notice that $|Z^c| = 0$. Fix $y \in Z$ arbitrarily. Then (A.6), (A.1), and Lebesgue's theorem imply

$$\begin{aligned}
\nu_n(y,y') := \int_0^{y+y'} f(y'')\beta_{c,n}(y+y',y'')\ dy'' \\
\longrightarrow \int_0^{y+y'} f(y'')\beta_c(y+y',y'')\ dy'' =: \nu(y,y')\ ,
\end{aligned}$$

for each $y' \in B_y$. Therefore,

$$\nu_n(y,\cdot)K_n(y,\cdot) \to \nu(y,\cdot)K(y,\cdot) \quad \text{a.e. on} \quad (0, y_0 - y) \tag{A.7}$$

in view of (A.2). Since

$$\|\nu_n(y,\cdot)K_n(y,\cdot)\|_{L_\infty((0,y_0-y))} \le 2\|f\|_\infty\|K\|_\infty\ , \quad n \in \mathbb{N}\ ,$$

we obtain from Proposition A.1 that $h_n(y) \to h(y)$ for $y \in Z$. Thereby, h is defined as h_n but with β_c, K, and w instead of $\beta_{c,n}$, K_n, and w_n, respectively. Again with the aid of Proposition A.1 we deduce from

$$\|h_n\|_{L_\infty(Y)} \leq 2\|f\|_\infty \, \|K\|_\infty c_0 \ , \quad n \in \mathbb{N} \ ,$$

that $I_n \to 0$. Similarly $II_n \to 0$ holds. Finally, an application of Lebesgue's theorem yields $III_n \to 0$. Thus (A.5) is valid. Based on (3.14) it is not difficult to modify the above proof to deduce $\tilde{L}_c^3(K_n)[w_n] \to L_c^3[w]$ in $L_{1,\mathrm{w}}$. □

Lastly, we state the convergence result for the scattering operator which can be proven analogously as the previous Lemma. Recall that

$$\begin{aligned}\tilde{L}_s(\beta_{s,n}, K_n)[v](y) := \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} &\beta_{s,n}(y',y)K_n(y'',y'-y'')v(y'')v(y'-y'')\ dy''dy' \\ &- \frac{1}{2}v(y)\int_{y_0-y}^{y_0}\int_{y+y'-y_0}^{y_0} \beta_{s,n}(y+y',y'')\ dy''\, K_n(y,y')v(y')\ dy' \ ,\end{aligned}$$

and that

$$\begin{aligned}L_s[v](y) := \frac{1}{2}\int_{y_0}^{y_0+y}\int_{y'-y_0}^{y_0} &\beta_s(y',y)K(y'',y'-y'')v(y'')v(y'-y'')\ dy''dy' \\ &- v(y)\int_{y_0-y}^{y_0} K(y,y')v(y')\ dy' \ .\end{aligned}$$

Lemma A.4. *The convergence* $\tilde{L}_s(K_n, \beta_{s,n})[w_n] \to L_s[w]$ *in* $L_{1,\mathrm{w}}$ *is valid.*

In particular, if $\gamma_n \equiv \gamma$, $\beta_{c,n} \equiv \beta_c$, $\beta_{s,n} \equiv \beta_s$, and $K_n \equiv K$ we have the following result.

Corollary A.5. *The map* $L[\cdot] = L_b[\cdot] + L_c[\cdot] + L_s[\cdot] : L_{1,\mathrm{w}} \to L_{1,\mathrm{w}}$ *is sequentially continuous.*

Part 2

Coalescence and Breakage Processes with Diffusion

4. Preliminaries

In part 1 we investigated the evolution of a liquid-liquid dispersion assuming droplets to be uniformly distributed. The aim of this part is to remove this fundamental assumption, allowing the droplet size distribution function to depend on spatial coordinates, and studying well-posedness of the underlying physical model. As we shall see, taking into account diffusion complicates the problem enormously and requires a different approach, of course.

Let $u = u(t,x,y)$ denote the distribution function of droplet size and consider the uncountable set of partial integro-differential equations

$$\begin{aligned} \partial_t u(y) - d(t,x,y)\Delta_x u(y) &= L(t,x,u)(y) && \text{in } \Omega\ , \quad t>0\ , \quad y \in Y\ , \\ \partial_\nu u(y) &= 0 && \text{on } \partial\Omega\ , \quad t>0\ , \quad y\in Y\ , \\ u(0,\cdot,y) &= u^0(y) && \text{in } \Omega\ , \quad y \in Y\ , \end{aligned} \tag{$**$}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded and smooth domain, ν is the outward normal vector of Ω and $Y = (0, y_0]$. Moreover, the 'reaction term' L is given by

$$L(t,x,u) := L_b(t,x,u) + L_c(t,x,u) + L_s(t,x,u)\ ,$$

where the operators L_b, L_c, and L_s are defined as in part 1 (see page 13) but with kernels $\gamma, \beta_c, \beta_s, K, P$, and Q now depending also on $(t,x) \in \mathbb{R}^+ \times \Omega$. In contrary to the previous part we neglect the efficiency factor $\varphi(u)$ for simplicity.

Our approach to Problem $(**)$ is to interpret it as a single equation. Formally, this is obtained by putting $A(t) := -d(t,\cdot,\cdot)\Delta$ with respect to Neumann boundary conditions, so that Problem $(**)$ can be rewritten as a vector-valued Cauchy Problem of the form

$$\begin{aligned} \dot{u} + A(t)u &= L(t,u)\ , \quad t>0\ , \\ u(0) &= u^0\ . \end{aligned} \tag{$(CP)_{u^0}$}$$

It turns out that — as in the scalar case — the operator $-A(t)$ is the generator of an analytic semigroup on $L_p(\Omega, E)$ with domain of definition

$$D\big(A(t)\big) \doteq H^2_{p,\mathcal{B}}(\Omega,E) := \big\{u \in H^2_p(\Omega,E)\,;\,\partial_\nu u = 0\big\}\ ,$$

where E is an appropriate function space over Y. A natural choice of this state space E would be $L_1(Y)$ which is, however, impossible as we shall see.
Once this formal reasoning is made rigorous, it is indispensable to have an exact characterization of the interpolation spaces between $L_p(\Omega,E)$ and $D\big(A(t)\big) \doteq H^2_{p,\mathcal{B}}(\Omega,E)$ in order to take full advantage of semigroup theory. This is the purpose of chapter 6. Bearing in mind the well-known interpolation results of Grisvard [**30**], [**31**], [**32**], and Seeley [**54**], [**55**] for the finite-dimensional case, one may ask on what conditions on the underlying Banach space E these results can be generalized to the infinite-dimensional setting. Our arguments are inspired by those of Guidetti [**33**] and require, regrettably, a Hilbert space leading to — at least from a physical point of view — the somehow artificial state space $E = L_2(Y)$.
In chapter 7, we first verify that the above operator $-A(t)$ indeed generates an analytic semigroup by using rather new results (cf. [**23**]) on maximal regularity of elliptic operators acting on $L_p(\Omega,E)$. From this we derive existence of a unique maximal solution for Problem $(**)$, which is non-negative and conserves the total mass. In special cases we obtain global existence.

5. Notations and Conventions

We briefly collect some basic spaces and their properties which we will use in the sequel. For more detailed information and proofs we refer in particular to [**9**], but also to [**2**], [**3**], and [**8**].

Let X and Z be locally convex spaces. We denote by $\mathcal{L}(X,Z)$ the set of all bounded linear operators from X into Z. We put $\mathcal{L}(X) := \mathcal{L}(X,X)$. Further, $\mathcal{L}is(X,Z)$ consists of all topological linear isomorphisms from X onto Z and $\mathcal{L}aut(X) := \mathcal{L}is(X,X)$. If X is a linear subspace of Z such that the natural injection $i : [x \mapsto x]$ belongs to $\mathcal{L}(X,Z)$, we express this by $X \hookrightarrow Z$. If this embedding is also dense, we write $X \overset{d}{\hookrightarrow} Z$, whereas $X \doteq Z$ means that $X \hookrightarrow Z$ and $Z \hookrightarrow X$.
For the remainder, let $E := (E, |\cdot|_E)$ be a Banach space and X a nonempty open subset of $\mathbb{R}^n$. We say that E is a *UMD space* if the Hilbert transform is a bounded operator on $L_p(\mathbb{R}, E)$ for some $p \in (1,\infty)$ (for a precise definition see [**8**, III.4.4]). Note that a UMD space is necessarily reflexive and that Hilbert spaces possess the UMD property.
$\mathcal{D}(X,E)$ is the space of all E-valued test functions, that is, the locally convex space of all smooth E-valued functions with compact supports in X, equipped with its inductive limit topology (as in the scalar-valued case) and $\mathcal{D}(X) := \mathcal{D}(X,\mathbb{R})$. We write $\mathcal{S}(\mathbb{R}^n, E)$ for the Schwartz space of all rapidly decreasing smooth E-valued functions on $\mathbb{R}^n$, endowed with its usual family of seminorms and we put $\mathcal{S}(\mathbb{R}^n) := \mathcal{S}(\mathbb{R}^n, \mathbb{R})$. Further, $\mathcal{D}'(X,E) := \mathcal{L}\big(\mathcal{D}(X), E\big)$ is the space of all E-valued distributions on X and $\mathcal{S}'(\mathbb{R}^n, E) := \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n), E\big)$ denotes the space of all E-valued tempered distributions on $\mathbb{R}^n$. Both of these spaces are endowed with the topology of uniform convergence on bounded subsets.
Recall that for $u \in \mathcal{D}'(X,E)$ and $\alpha \in \mathbb{N}^n$ the distributional derivative $\partial^\alpha u$ is defined by

$$(\partial^\alpha u)(\varphi) := (-1)^{|\alpha|} u(\partial^\alpha \varphi) \ , \quad \varphi \in \mathcal{D}(X) \ .$$

By $\mathcal{F} \in \mathcal{L}aut\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \cap \mathcal{L}aut\big(\mathcal{S}(\mathbb{R}^n, E)\big)$ we denote the Fourier transform, and occasionally we put $\widehat{u} := \mathcal{F}u$ for $u \in \mathcal{S}'(\mathbb{R}^n, E)$.
Whenever it makes sense we mean by $f * g$ the (eventually vector-valued) convolution of f and g.
If $a \in C^\infty(\mathbb{R}^n, E)$ and if for any given $\alpha \in \mathbb{N}^n$ there exist $m_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$|\partial^\alpha a(x)|_E \le c_\alpha \big(1 + |x|^2\big)^{m_\alpha} \ , \quad x \in \mathbb{R}^n \ ,$$

we say that a belongs to the space of slowly increasing smooth functions $\mathcal{O}_M(\mathbb{R}^n, E)$. If $a \in \mathcal{O}_M(\mathbb{R}^n) := \mathcal{O}_M(\mathbb{R}^n, \mathbb{R})$, then

$$[\varphi \mapsto a\varphi] \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big) \ .$$

Hence, given $a \in \mathcal{O}_M(\mathbb{R}^n)$,

$$a(D) := \mathcal{F}^{-1} a \mathcal{F} \in \mathcal{L}\big(\mathcal{S}'(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(\mathcal{S}(\mathbb{R}^n, E)\big) \ .$$

For $s \in \mathbb{R}^+$, $BUC^s(X,E)$ is the Banach space of all functions $u : X \to E$ whose derivatives of orders at most $[s]$ are bounded and uniformly continuous and whose derivatives of order $[s]$ are $(s-[s])$-Hölder continuous if $s \notin \mathbb{N}$.
The *Sobolev space* $W_p^m(X,E)$ for $m \in \mathbb{N}$ and $p \in [1,\infty]$ is the Banach space consisting of all $u \in L_p(X,E)$ such that the (distributional) derivative $\partial^\alpha u$ belongs to $L_p(X,E)$ for all

$|\alpha| \leq m$, equipped with its obvious norm.
If $0 < \theta < 1$ and $1 \leq p < \infty$, we put

$$[u]_{\theta,p} := \left(\int_{X \times X} \left(\frac{|u(x) - u(y)|_E}{|x-y|^\theta} \right)^p \frac{d(x,y)}{|x-y|^n} \right)^{1/p} .$$

Then we define for $m \in \mathbb{N}$ and $0 < \theta < 1$ the vector-valued *Slobodeckii space*

$$W_p^{m+\theta}(X,E) := \left(\left\{ u \in W_p^m(X,E) \, ; \, \|u\|_{m+\theta,p} < \infty \right\}, \| \cdot \|_{m+\theta,p} \right) ,$$

where

$$\|u\|_{m+\theta,p} := \|u\|_{W_p^m(X,E)} + \max_{|\alpha|=m} [\partial^\alpha u]_{\theta,p} .$$

If $m \in \dot{\mathbb{N}}$ and $0 \leq \theta < 1$ then $W_p^{-m+\theta}(X,E)$ is the Banach space of all E-valued distributions u on X having a representation

$$u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha u_\alpha$$

with $u_\alpha \in W_p^\theta(X,E)$, equipped with the norm

$$u \mapsto \inf \left(\sum_{|\alpha| \leq m} \|u_\alpha\|_{\theta,p} \right) ,$$

the infimum being taken over all such representations. It holds

$$W_p^s(X,E) \hookrightarrow W_p^t(X,E) , \quad s \geq t , \quad 1 \leq p < \infty .$$

Fix $\psi \in \mathcal{D}(\mathbb{R}^n)$ with $0 \leq \psi \leq 1$ and $\psi|_{\mathbb{B}^n} = 1$ as well as $\operatorname{supp} \psi \subset 2\mathbb{B}^n$. Then set $\psi_0 := \psi$ and $\psi_k := \psi(2^{-k} \cdot) - \psi(2^{-k+1} \cdot)$ for $k \in \dot{\mathbb{N}}$, so that $(\psi_k)_{k \in \mathbb{N}}$ is a dyadic resolution of the identity on $\mathbb{R}^n$. For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, the *Besov space* $B_{p,q}^s(\mathbb{R}^n, E)$ is defined as the set of all $u \in \mathcal{S}'(\mathbb{R}^n, E)$ satisfying

$$\|u\|_{B_{p,q}^s} := \|u\|_{B_{p,q}^s(\mathbb{R}^n,E)} := \left\| \left(2^{sk} \|\psi_k(D)u\|_{L_p(\mathbb{R}^n,E)} \right)_{k \in \mathbb{N}} \right\|_{l_q} < \infty .$$

It is a Banach space with the norm $\| \cdot \|_{B_{p,q}^s}$ and different choices of ψ lead to equivalent norms. Then $\mathcal{S}(\mathbb{R}^n, E)$ is dense in $B_{p,q}^s(\mathbb{R}^n, E)$ provided $p, q < \infty$. For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ one has

$$B_{p,q_1}^s(\mathbb{R}^n, E) \hookrightarrow B_{p,q_0}^s(\mathbb{R}^n, E) , \quad 1 \leq q_1 \leq q_0 \leq \infty . \tag{5.1}$$

Moreover, for $1 \leq p, q_0, q_1 \leq \infty$ one has

$$B_{p,q_1}^{s_1}(\mathbb{R}^n, E) \hookrightarrow B_{p,q_0}^{s_0}(\mathbb{R}^n, E) , \quad s_1 > s_0 , \tag{5.2}$$

as well as for $1 \leq p_0, p_1, q \leq \infty$

$$B_{p_1,q}^{s_1}(\mathbb{R}^n, E) \hookrightarrow B_{p_0,q}^{s_0}(\mathbb{R}^n, E) , \quad p_1 < p_0 , \quad s_1 - n/p_1 \geq s_0 - n/p_0 . \tag{5.3}$$

Also,

$$B_{p,p}^s(\mathbb{R}^n, E) \doteq W_p^s(\mathbb{R}^n, E) , \quad s \in \mathbb{R} \setminus \mathbb{Z} , \quad 1 \leq p < \infty , \tag{5.4}$$

and

$$B_{\infty,\infty}^s(\mathbb{R}^n, E) \doteq BUC^s(\mathbb{R}^n, E) , \quad s \in \mathbb{R}^+ \setminus \mathbb{N} . \tag{5.5}$$

Denoting for $0 < \theta < 1$ and $1 \le q \le \infty$ by $(\cdot,\cdot)_{\theta,q}$ the real interpolation functor of exponent θ and parameter q and by $[\cdot,\cdot]_\theta$ the complex interpolation functor of exponent θ (see section 6.3 for precise definitions), it holds for $1 \le p, q, q_0, q_1 \le \infty$ and $s_0 \ne s_1$

$$\big(B^{s_0}_{p,q_0}(\mathbb{R}^n, E), B^{s_1}_{p,q_1}(\mathbb{R}^n, E)\big)_{\theta,q} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n, E) \ , \tag{5.6}$$

and

$$\big[B^{s_0}_{p,q}(\mathbb{R}^n, E), B^{s_1}_{p,q}(\mathbb{R}^n, E)\big]_{\theta} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n, E) \ , \quad q < \infty \ . \tag{5.7}$$

Moreover, $\mathcal{F}^{-1}(1+|\xi|^2)^{s/2}\mathcal{F} \in \mathcal{L}is\big(B^t_{p,q}(\mathbb{R}^n, E), B^{t-s}_{p,q}(\mathbb{R}^n, E)\big)$ for $t, s \in \mathbb{R}$.
Let $s \in \mathbb{R}$ and $1 \le p < \infty$. Then $H^s_p(\mathbb{R}^n, E)$ is the *Bessel potential space* defined by

$$H^s_p(\mathbb{R}^n, E) := \Big(\big\{u \in \mathcal{S}'(\mathbb{R}^n, E)\, ;\, \mathcal{F}^{-1}\big(1+|\xi|^2\big)^{s/2}\mathcal{F}u \in L_p(\mathbb{R}^n, E)\big\}\, ,\, \|\cdot\|_{H^s_p}\Big) \ ,$$

where

$$\|u\|_{H^s_p} := \|u\|_{H^s_p(\mathbb{R}^n,E)} := \big\|\mathcal{F}^{-1}\big(1+|\xi|^2\big)^{s/2}\mathcal{F}u\big\|_{L_p(\mathbb{R}^n,E)} \ .$$

It is a Banach space with the norm $\|\cdot\|_{H^s_p}$ and $\mathcal{S}(\mathbb{R}^n, E)$ is dense in $H^s_p(\mathbb{R}^n, E)$. Furthermore,

$$H^s_p(\mathbb{R}^n, E) \overset{d}{\hookrightarrow} H^t_p(\mathbb{R}^n, E) \ , \quad s > t \ , \quad 1 \le p < \infty \ . \tag{5.8}$$

For the remainder let E be a UMD space and $1 < p < \infty$. Then

$$H^m_p(\mathbb{R}^n, E) \doteq W^m_p(\mathbb{R}^n, E) \ , \quad m \in \mathbb{Z} \ , \tag{5.9}$$

and, for $s \in \mathbb{R}$, the dual space $\big[H^s_p(\mathbb{R}^n, E)\big]'$ of $H^s_p(\mathbb{R}^n, E)$ with respect to the duality pairing induced by the L_p-duality pairing coincides with $H^{-s}_{p'}(\mathbb{R}^n, E')$ (with equivalent norms). Here we use the convention $1 = 1/p + 1/p'$. Further, for $0 < \theta < 1$ and $s_i \in \mathbb{R}$, it holds

$$\big[H^{s_0}_p(\mathbb{R}^n, E), H^{s_1}_p(\mathbb{R}^n, E)\big]_{\theta} \doteq H^{(1-\theta)s_0+\theta s_1}_p(\mathbb{R}^n, E) \ , \tag{5.10}$$

and for $1 \le q \le \infty$

$$\big(H^{s_0}_p(\mathbb{R}^n, E), H^{s_1}_p(\mathbb{R}^n, E)\big)_{\theta,q} \doteq B^{(1-\theta)s_0+\theta s_1}_{p,q}(\mathbb{R}^n, E) \ . \tag{5.11}$$

For $\alpha \in \mathbb{N}^n$, $\partial^\alpha \in \mathcal{L}\big(H^s_p(\mathbb{R}^n, E), H^{s-|\alpha|}_p(\mathbb{R}^n, E)\big)$. Finally, u belongs to $H^s_p(\mathbb{R}^n, E)$ iff $\partial^\alpha u \in H^{s-m}_p(\mathbb{R}^n, E)$ for all $|\alpha| \le m \in \dot{\mathbb{N}}$, and

$$\Big[u \mapsto \sum_{|\alpha|\le m} \|\partial^\alpha u\|_{H^{s-m}_p(\mathbb{R}^n,E)}\Big] \tag{5.12}$$

is an equivalent norm on $H^s_p(\mathbb{R}^n, E)$.

6. On Interpolation with Boundary Conditions

The aim of this chapter is to prove the interpolation formula

$$\big[L_p(\Omega,E),H^2_{p,\mathcal{B}}(\Omega,E)\big]_\theta \doteq H^{2\theta}_{p,\mathcal{B}}(\Omega,E)\ ,\quad 0<\theta<1\ ,\quad 2\theta\neq 1+1/p\ , \tag{6.1}$$

where E is a Hilbert space and — roughly speaking — $H^s_{p,\mathcal{B}}(\Omega,E)$ consists of all those $u\in H^s_p(\Omega,E)$ satisfying $\partial_\nu u=0$. Our arguments are very similar to those of Guidetti [**33**] who proved the analogue result for scalar-valued Besov spaces (see section 6.4). We begin with some auxiliary results.

6.1. A Multiplier Result

Let $\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times\dot{\mathbb{R}}^+$ be the upper half space. Provided E is a Hilbert space, we prove in this section that the characteristic function $\chi_{\mathbb{R}^n_+}$ is a multiplier for the Bessel potential spaces $H^\alpha_p(\mathbb{R}^n,E)$ if $1<p<\infty$ and $-1+1/p<\alpha<1/p$, that is, there exists some $c_{\alpha,p}>0$ such that

$$\|\chi_{\mathbb{R}^n_+}u\|_{H^\alpha_p(\mathbb{R}^n,E)}\le c_{\alpha,p}\|u\|_{H^\alpha_p(\mathbb{R}^n,E)}\ ,\quad u\in H^\alpha_p(\mathbb{R}^n,E)\ .$$

The scalar-valued version of this result has been obtained by Strichartz [**61**] (see also [**41**] for $p=2$), whose arguments we will generalize to the vector-valued case. Although most of the ideas used in the proof can be adapted to arbitrary Banach spaces possessing the UMD property, its main idea — namely, to work with an equivalent norm on $H^\alpha_p(\mathbb{R}^n,E)$ which can be handled easier (cf. Proposition 6.7) — is restricted to Hilbert spaces. Nevertheless, since some of the subsequent auxiliary results (of this but also of the following section) are interesting in themselves, we state them for UMD spaces where possible.

For an arbitrary Banach space $X:=\big(X,|\cdot|_X\big)$ and $1\le p<\infty$ define the *Marcinkiewicz space* $L^\star_p(\mathbb{R}^n,X)$ by

$$L^\star_p(\mathbb{R}^n,X) := \Big(\big\{f\in L_1(\mathbb{R}^n,X)+L_\infty(\mathbb{R}^n,X)\,;\,\|f\|_{L^\star_p(\mathbb{R}^n,X)}<\infty\big\}\,,\,\|\cdot\|_{L^\star_p(\mathbb{R}^n,X)}\Big)$$

where

$$\|f\|_{L^\star_p(\mathbb{R}^n,X)} := \sup_{\sigma>0}\ \sigma\Big(\lambda_n\big(\{z\in\mathbb{R}^n\,;\,|f(z)|_X>\sigma\}\big)\Big)^{1/p}\ .$$

Here, λ_n denotes Lebesgue's measure on $\mathbb{R}^n$. Then $L^\star_p(\mathbb{R}^n,X)$ is a quasi-Banach space (see [**62**, §1.18.6] and [**35**, §1.3] for details). For brevity put $L^\star_p(\mathbb{R}^n):=L^\star_p(\mathbb{R}^n,\mathbb{R})$.

For the proof of the following Marcinkiewicz interpolation theorem we refer to [**62**, Thm.1.18.7/2] or [**35**, Thm.5.3.2].

Proposition 6.1. *Let X_1 and X_2 be Banach spaces and $p_i,q_i\in(1,\infty)$ with $p_0\neq p_1$ and $q_0\neq q_1$. For $0<\theta<1$ define p and q by*

$$\frac{1}{p}:=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\ ,\qquad \frac{1}{q}:=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}\ .$$

(i) If T is a linear operator satisfying for some $r_i>0$

$$\|Tf\|_{L^\star_{q_i}(\mathbb{R}^n,X_2)}\le r_i\|f\|_{L_{p_i}(\mathbb{R}^n,X_1)}\ ,\quad i=0,1\ ,$$

then

$$\|Tf\|_{L_q(\mathbb{R}^n,X_2)} \le c(\theta) r_0^{1-\theta} r_1^{\theta} \|f\|_{L_p(\mathbb{R}^n,X_1)} \quad \textit{if} \;\; p \le q \ .$$

(ii) If T is a linear operator satisfying for some $r_i > 0$

$$\|Tf\|_{L_{q_i}(\mathbb{R}^n,X_2)} \le r_i \|f\|_{L_{p_i}(\mathbb{R}^n,X_1)} \ , \quad i = 0,1 \ ,$$

then

$$\|Tf\|_{L_q^\star(\mathbb{R}^n,X_2)} \le c(\theta) r_0^{1-\theta} r_1^{\theta} \|f\|_{L_p^\star(\mathbb{R}^n,X_1)} \ .$$

In order to give another characterization of $H_p^\alpha(\mathbb{R}^n, E)$, define for $\alpha > 0$ the *Bessel potential* G_α by

$$G_\alpha(x) := \frac{1}{\Gamma\big(\frac{\alpha}{2}\big)(4\pi)^{n/2}} \int_0^\infty e^{-\frac{|x|^2}{4t}} t^{-\frac{n}{2}+\frac{\alpha}{2}} e^{-t} \frac{dt}{t} \ , \quad x \in \mathbb{R}^n \ ,$$

where Γ is the gamma function.

Lemma 6.2. *It holds $G_\alpha \in L_1(\mathbb{R}^n)$ and $G_\alpha = \mathcal{F}^{-1}\big(1+|\cdot|^2\big)^{-\alpha/2}$ for $\alpha > 0$. Furthermore, $G_\alpha \in L^\star_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ if $0 < \alpha < n$.*

Proof. For the first and the second assertion we refer to [**3**, Lem.3.3.1]. If $0 < \alpha < n$ use the substitution $s := |x|^2/4t$ in the definition of $G_\alpha(x)$ to deduce

$$0 \le G_\alpha(x) \le c_{n,\alpha} |x|^{-n+\alpha} \ , \quad x \in \mathbb{R}^n \ ,$$

and whence $G_\alpha \in L^\star_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$. □

The following corollary is a consequence of the convolution theorem (see [**9**, Thm.3.6]).

Corollary 6.3. *Let E be a Banach space, $1 \le p < \infty$, and $\alpha > 0$. Then $H_p^\alpha(\mathbb{R}^n, E) = \big\{G_\alpha * g \, ; \, g \in L_p(\mathbb{R}^n, E)\big\}$ and $\|G_\alpha * g\|_{H_p^\alpha(\mathbb{R}^n,E)} = \|g\|_{L_p(\mathbb{R}^n,E)}$.*

In the sequel we use the following notation. Given any (quasi-) normed spaces X, Y, and Z we mean by writing $X \bullet Y \hookrightarrow Z$ [resp. $X * Y \hookrightarrow Z$] that multiplication [resp. convolution] $X \times Y \to Z$ is continuous. For instance, if E is a Banach space and $1 \le s, t, r \le \infty$ then Hölder's inequality gives

$$L_s(\mathbb{R}^n) \bullet L_t(\mathbb{R}^n, E) \hookrightarrow L_r(\mathbb{R}^n, E) \ , \quad \frac{1}{r} = \frac{1}{s} + \frac{1}{t} \ , \tag{6.2}$$

while Young's inequality implies

$$L_s(\mathbb{R}^n) * L_t(\mathbb{R}^n, E) \hookrightarrow L_r(\mathbb{R}^n, E) \ , \quad 1 + \frac{1}{r} = \frac{1}{s} + \frac{1}{t} \ . \tag{6.3}$$

Proposition 6.4. *Let E be a Banach space and suppose that $\alpha > 0$ and $1 < p < n/\alpha$. Then $L^\star_{n/\alpha}(\mathbb{R}^n) \bullet H_p^\alpha(\mathbb{R}^n, E) \hookrightarrow L_p(\mathbb{R}^n, E)$.*

Proof. Owing to Lemma 6.2 and Corollary 6.3 it suffices to show that

$$\|\psi(G_\alpha * g)\|_{L_p(\mathbb{R}^n,E)} \le c \|\psi\|_{L^\star_{n/\alpha}(\mathbb{R}^n)} \, \|G_\alpha\|_{L^\star_{\frac{n}{n-\alpha}}(\mathbb{R}^n)} \, \|g\|_{L_p(\mathbb{R}^n,E)} \tag{6.4}$$

for all $\psi \in L^\star_{n/\alpha}(\mathbb{R}^n)$ and all $g \in L_p(\mathbb{R}^n, E)$. For, fix $\alpha_i > 0$ with $p < n/\alpha_0 < n/\alpha < n/\alpha_1$ and put $1/q_i := 1/p - \alpha_i/n$ for $i = 0, 1$. By (6.3) we have

$$L_{\frac{n}{n-\alpha_i}}(\mathbb{R}^n) * L_p(\mathbb{R}^n, E) \hookrightarrow L_{q_i}(\mathbb{R}^n, E) \ , \quad i = 0,1 \ ,$$

and thus, in view of Proposition 6.1(ii),

$$L^{\star}_{\frac{n}{n-\alpha}}(\mathbb{R}^n) * L_p(\mathbb{R}^n, E) \hookrightarrow L^{\star}_q(\mathbb{R}^n, E) \ , \quad p < \frac{n}{\alpha} \ , \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \ .$$

Similarly, we deduce from this and Proposition 6.1(i)

$$L^{\star}_{\frac{n}{n-\alpha}}(\mathbb{R}^n) * L_p(\mathbb{R}^n, E) \hookrightarrow L_q(\mathbb{R}^n, E) \ , \quad p < \frac{n}{\alpha} \ , \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \ . \tag{6.5}$$

On the other hand, (6.2) yields

$$L_{n/\alpha}(\mathbb{R}^n) \bullet L_q(\mathbb{R}^n, E) \hookrightarrow L_p(\mathbb{R}^n, E) \ , \quad p < \frac{n}{\alpha} \ , \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \ ,$$

from which it follows by Proposition 6.1(ii)

$$L^{\star}_{n/\alpha}(\mathbb{R}^n) \bullet L_q(\mathbb{R}^n, E) \hookrightarrow L^{\star}_p(\mathbb{R}^n, E) \ , \quad p < \frac{n}{\alpha} \ , \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \ .$$

Taking into account (6.5), we conclude

$$L^{\star}_{n/\alpha}(\mathbb{R}^n) \bullet \big(L^{\star}_{\frac{n}{n-\alpha}}(\mathbb{R}^n) * L_p(\mathbb{R}^n, E) \big) \hookrightarrow L^{\star}_p(\mathbb{R}^n, E) \ , \quad 1 < p < \frac{n}{\alpha} \ .$$

Finally, applying Proposition 6.1(i) once more we obtain

$$L^{\star}_{n/\alpha}(\mathbb{R}^n) \bullet \big(L^{\star}_{\frac{n}{n-\alpha}}(\mathbb{R}^n) * L_p(\mathbb{R}^n, E) \big) \hookrightarrow L_p(\mathbb{R}^n, E) \ , \quad 1 < p < \frac{n}{\alpha} \ ,$$

and whence (6.4). □

As in [**61**] for the scalar-valued case, we generalize Fubini's theorem to Hilbert-space-valued Bessel potential spaces.
Recall that $m \in L_\infty(\mathbb{R}^n)$ is said to be a *Fourier multiplier for* $L_p(\mathbb{R}^n, E)$ if

$$\|\mathcal{F}^{-1} m \mathcal{F} u\|_{L_p(\mathbb{R}^n,E)} \le c \|u\|_{L_p(\mathbb{R}^n,E)} \ , \quad u \in L_p(\mathbb{R}^n, E) \ .$$

For $u : \mathbb{R}^n \to E$ and $1 \le k \le n$ put

$$u_k(x') := u(x'_1, \ldots, x'_{k-1}, \cdot, x'_k, \ldots, x'_{n-1}) \ , \quad x' \in \mathbb{R}^{n-1} \ .$$

Proposition 6.5. *Let E be a Hilbert space and suppose that $1 < p < \infty$ and $\alpha > 0$. Then $u \in L_p(\mathbb{R}^n, E)$ belongs to $H^\alpha_p(\mathbb{R}^n, E)$ if and only if for each $k \in \{1, \ldots, n\}$ the map $\big[x' \mapsto \|u_k(x')\|_{H^\alpha_p(\mathbb{R},E)}\big]$ belongs to $L_p(\mathbb{R}^{n-1})$ and then*

$$\|u\|_{H^\alpha_p(\mathbb{R}^n,E)} \approx \sum_{k=1}^{n} \big\| \, \|u_k\|_{H^\alpha_p(\mathbb{R},E)} \, \big\|_{L_p(\mathbb{R}^{n-1})} \ .$$

Proof. Let $1 \le k \le n$ be arbitrary and define

$$a_k(\xi) := \big(1 + \xi_k^2\big)^{\alpha/2} \big(1 + |\xi|^2\big)^{-\alpha/2} \ , \quad \xi \in \mathbb{R}^n \ .$$

Then, for $\beta \in \mathbb{N}^n$ with $\beta \le (1, \ldots, 1)$, there exists $c_\beta > 0$ such that

$$|\xi^\beta \partial^\beta a_k(\xi)| \le c_\beta \ , \quad \xi \in \mathbb{R}^n \ .$$

Since E is a Hilbert space, the vector-valued Mikhlin theorem of [**71**, Prop.3] entails that a_k is a Fourier multiplier for $L_p(\mathbb{R}^n, E)$. Similarly, the function b defined by

$$b(\xi) := \big(1 + |\xi|^2\big)^{\alpha/2} \Big[\sum_{j=1}^{n} \big(1 + \xi_j^2\big)^{\alpha/2} \Big]^{-1} \ , \quad \xi \in \mathbb{R}^n \ ,$$

is a Fourier multiplier for $L_p(\mathbb{R}^n, E)$. Thus,

$$w \mapsto \sum_{k=1}^{n} \|\mathcal{F}^{-1}\big(1+\xi_k^2\big)^{\alpha/2}\mathcal{F}w\|_{L_p(\mathbb{R}^n,E)}$$

is an equivalent norm for $H_p^\alpha(\mathbb{R}^n, E)$. Temporarily, define for $1 \le k \le n$

$$\mathcal{F}_k\varphi(x_1,\ldots,x_{k-1},\xi_k,x_{k+1},\ldots,x_n) := \int_{\mathbb{R}} e^{-ix_k\xi_k}\varphi(x)\ dx_k\ , \quad \varphi \in \mathcal{S}(\mathbb{R}^n, E)\ ,$$

and extend $\mathcal{F}_k$ to $\mathcal{S}'(\mathbb{R}^n, E)$ by putting

$$\mathcal{F}_k w(\varphi) := w(\mathcal{F}_k\varphi)\ , \quad \varphi \in \mathcal{S}(\mathbb{R}^n)\ , \quad w \in \mathcal{S}'(\mathbb{R}^n, E)\ .$$

Likewise, $\mathcal{F}_k^{-1}$, defined by

$$\mathcal{F}_k^{-1}\varphi(x_1,\ldots,x_{k-1},\xi_k,x_{k+1},\ldots,x_n) := \frac{1}{2\pi}\mathcal{F}_k\varphi(x_1,\ldots,x_{k-1},-\xi_k,x_{k+1},\ldots,x_n)\ ,$$

for $\varphi \in \mathcal{S}(\mathbb{R}^n, E)$, is extended to $\mathcal{S}'(\mathbb{R}^n, E)$. The assertion is then a consequence of the facts that

$$\mathcal{F}^{-1}\big(1+\xi_k^2\big)^{\alpha/2}\mathcal{F}u = \mathcal{F}_k^{-1}\big(1+\xi_k^2\big)^{\alpha/2}\mathcal{F}_k u\ , \quad u \in \mathcal{S}'(\mathbb{R}^n, E)\ , \quad 1 \le k \le n\ ,$$

and $L_p(\mathbb{R}^n, E) = L_p\big(\mathbb{R}^{n-1}, L_p(\mathbb{R}, E)\big)$. □

Remark 6.6. For simplicity we assumed E to be a Hilbert space in the previous proposition although less is required. Actually, the proposition is valid provided E is a UMD space possessing a *local unconditional structure* (see [**71**] for a definition) as L_q-spaces, for instance. On the other hand, the arguments used in the proof are false for arbitrary UMD spaces. Indeed, defining a_k as in the above proof, the function $\xi \mapsto |\xi|^{|\beta|}\,|\partial^\beta a_k(\xi)|$ for $\beta \in \mathbb{N}^n$ with $\beta \le (1,\ldots,1)$ remains not bounded as $|\xi| \to \infty$ in general. Hence, the vector-valued Mikhlin theorem [**71**, Prop.3] for arbitrary UMD spaces cannot be applied.

Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{R}^n$. Formally, define for $u : \mathbb{R}^n \to E$ and $\alpha > 0$

$$S_\alpha u(x) := \left(\int_0^\infty \left(\int_{\mathbb{B}^n} |u(x+ty)-u(x)|_E\ dy\right)^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2}\ , \quad x \in \mathbb{R}^n\ .$$

Proposition 6.7. *Let E be a Hilbert space. Suppose that $1 < p < \infty$ and $0 < \alpha < 1$. Then $u \in H_p^\alpha(\mathbb{R}^n, E)$ if and only if $u \in L_p(\mathbb{R}^n, E)$ and $S_\alpha u \in L_p(\mathbb{R}^n)$ and then*

$$\|u\|_{H_p^\alpha(\mathbb{R}^n,E)} \approx \|u\|_{L_p(\mathbb{R}^n,E)} + \|S_\alpha u\|_{L_p(\mathbb{R}^n)}\ .$$

Proof. Since E is a Hilbert space, this follows from [**53**, Rem.6, Prop.8] and the references therein. □

Remark 6.8. It seems to be worthwhile emphasizing that it is mainly this proposition which prevents work with more general spaces in the sequel, since it is true *only if* E is a Hilbert space.

Proposition 6.9. *Let E be a Hilbert space and assume that $1 < p < \infty$ and $0 < \alpha < 1$. Then $f \in L_\infty(\mathbb{R}^n)$ is a multiplier for $H_p^\alpha(\mathbb{R}^n, E)$, i.e*

$$\|fu\|_{H_p^\alpha(\mathbb{R}^n,E)} \le c\|u\|_{H_p^\alpha(\mathbb{R}^n,E)}\ , \quad u \in H_p^\alpha(\mathbb{R}^n, E)\ ,$$

if and only if

$$\|uS_\alpha f\|_{L_p(\mathbb{R}^n,E)} \le c\|u\|_{H_p^\alpha(\mathbb{R}^n,E)}\ , \quad u \in H_p^\alpha(\mathbb{R}^n, E)\ .$$

PROOF. Assume that $f \in L_\infty(\mathbb{R}^n)$ is a multiplier for $H_p^\alpha(\mathbb{R}^n, E)$ and let $u \in H_p^\alpha(\mathbb{R}^n, E)$ be arbitrary. Then

$$|u(x)|_E S_\alpha f(x) \le 2\|f\|_\infty S_\alpha u(x) + 2S_\alpha(uf)(x) , \quad \text{a.a. } x \in \mathbb{R}^n ,$$

so that according to Proposition 6.7

$$\|uS_\alpha f\|_{L_p(\mathbb{R}^n,E)} \le 2\|f\|_\infty \|S_\alpha u\|_{L_p(\mathbb{R}^n)} + 2\|S_\alpha(uf)\|_{L_p(\mathbb{R}^n)} \le c\|u\|_{H_p^\alpha(\mathbb{R}^n,E)} .$$

Conversely, suppose that $f \in L_\infty(\mathbb{R}^n)$ satisfies

$$\|uS_\alpha f\|_{L_p(\mathbb{R}^n,E)} \le c\|u\|_{H_p^\alpha(\mathbb{R}^n,E)} , \quad u \in H_p^\alpha(\mathbb{R}^n, E) .$$

Since

$$S_\alpha(uf)(x) \le 2\|f\|_\infty S_\alpha u(x) + 2|u(x)|_E S_\alpha f(x) , \quad \text{a.a. } x \in \mathbb{R}^n ,$$

one easily deduces, by virtue of Proposition 6.7, that f is a multiplier for $H_p^\alpha(\mathbb{R}^n, E)$. □

Corollary 6.10. *Let E be a Hilbert space and $0 \le \alpha < 1/p < 1$. Then $\chi_{(0,\infty)}$ is a multiplier for $H_p^\alpha(\mathbb{R}, E)$.*

PROOF. Provided $\alpha > 0$ it is not difficult to check that

$$S_\alpha \chi_{(0,\infty)}(x) \le c|x|^{-\alpha} , \quad x \in \mathbb{R} ,$$

for some constant $c > 0$. Hence $S_\alpha \chi_{(0,\infty)} \in L_{1/\alpha}^\star(\mathbb{R})$, and Proposition 6.4 yields

$$\|uS_\alpha \chi_{(0,\infty)}\|_{L_p(\mathbb{R},E)} \le c\|S_\alpha \chi_{(0,\infty)}\|_{L_{1/\alpha}^\star(\mathbb{R})} \|u\|_{H_p^\alpha(\mathbb{R},E)} , \quad u \in H_p^\alpha(\mathbb{R}, E) .$$

□

After these preparations we can establish now one of the main ingredients for the proof of the interpolation result (6.1). Recall that $1/p + 1/p' = 1$.

Theorem 6.11. *Assume that E is a Hilbert space and denote for $m \in \{1, \ldots, n\}$ by $\chi_m := \chi_{W^m}$ the characteristic function of $W^m := \mathbb{R}^{n-m} \times (\dot{\mathbb{R}}^+)^m$. Then χ_m is a multiplier for $H_p^\alpha(\mathbb{R}^n, E)$ provided $1 < p < \infty$ and $-1/p' < \alpha < 1/p$.*

PROOF. (i) Assume that $0 < \alpha < 1/p$. Since

$$\chi_m(x) = \prod_{j=n-m+1}^{n} \chi_{(0,\infty)}(x_j) , \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n ,$$

we obtain from Proposition 6.5 and Corollary 6.10

$$\|\chi_m u\|_{H_p^\alpha(\mathbb{R}^n,E)} \le c\|u\|_{H_p^\alpha(\mathbb{R}^n,E)} , \quad u \in H_p^\alpha(\mathbb{R}^n, E) .$$

(ii) Suppose now that $0 < \alpha < 1/p'$ and let $u \in \mathcal{D}(\mathbb{R}^n, E)$. Then

$$\chi_m u \in L_p(\mathbb{R}^n, E) \hookrightarrow H_p^{-\alpha}(\mathbb{R}^n, E) \doteq \big[H_{p'}^\alpha(\mathbb{R}^n, E)\big]'$$

and

$$\langle \chi_m u, \varphi\rangle_{H_{p'}^\alpha(\mathbb{R}^n,E)} = \langle u, \chi_m \varphi\rangle_{H_{p'}^\alpha(\mathbb{R}^n,E)} , \quad \varphi \in \mathcal{D}(\mathbb{R}^n, E) .$$

From (i) we then get

$$|\langle \chi_m u, \varphi\rangle_{H_{p'}^\alpha(\mathbb{R}^n,E)}| \le c\|u\|_{H_p^{-\alpha}(\mathbb{R}^n,E)} \|\varphi\|_{H_{p'}^\alpha(\mathbb{R}^n,E)} , \quad \varphi \in H_{p'}^\alpha(\mathbb{R}^n, E) ,$$

since $\mathcal{D}(\mathbb{R}^n, E)$ is dense in $H_{p'}^\alpha(\mathbb{R}^n, E)$, and whence

$$\|\chi_m u\|_{H_p^{-\alpha}(\mathbb{R}^n,E)} \le c\|u\|_{H_p^{-\alpha}(\mathbb{R}^n,E)} , \quad u \in \mathcal{D}(\mathbb{R}^n, E) . \tag{6.6}$$

The density of $\mathcal{D}(\mathbb{R}^n, E)$ in $H_p^{-\alpha}(\mathbb{R}^n, E)$ entails that we may extend χ_m in order to obtain (6.6) for all $u \in H_p^{-\alpha}(\mathbb{R}^n, E)$. □

6.2. Spaces on Domains and Traces

Throughout this section, we assume that E is a UMD space and that $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain.

If X is a nonempty open subset of $\mathbb{R}^n$, we denote by $r_X \in \mathcal{L}\big(\mathcal{D}'(\mathbb{R}^n, E), \mathcal{D}'(X, E)\big)$ the restriction operator for X, that is,

$$(r_X u)(\varphi) := u(\varphi) \ , \quad \varphi \in D(X) \ , \quad u \in \mathcal{D}'(\mathbb{R}^n, E) \ .$$

If $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define $S(X, E) := r_X S(\mathbb{R}^n, E)$ for $S \in \big\{H_p^s, B_{p,q}^s\big\}$ and we equip these spaces with the quotient space topology, i.e.

$$\|u\|_{S(X,E)} := \inf \big\{ \|\tilde{u}\|_{S(\mathbb{R}^n,E)} \, ; \, \tilde{u} \in S(\mathbb{R}^n, E), \, r_X \tilde{u} = u \big\} \ .$$

Then, these are Banach spaces. Furthermore, we set $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \dot{\mathbb{R}}^+$ and $\mathbb{R}^0 := \{0\}$.

Proposition 6.12. *Let $X \in \big\{\mathbb{R}^n_+, \Omega\big\}$ and $s \in \mathbb{R}$.*
(a) The restriction operator r_X is a retraction

- *(i) from $W_p^s(\mathbb{R}^n, E)$ onto $W_p^s(X, E)$ if $1 \leq p < \infty$,*
- *(ii) from $B_{p,q}^s(\mathbb{R}^n, E)$ onto $B_{p,q}^s(X, E)$ if $1 \leq p < \infty$ and $1 \leq q \leq \infty$,*
- *(iii) from $BUC^s(\mathbb{R}^n, E)$ onto $BUC^s(X, E)$ if $s \in \mathbb{R}^+$,*
- *(iv) from $H_p^s(\mathbb{R}^n, E)$ onto $H_p^s(X, E)$ if $1 < p < \infty$.*

Moreover, there exists a universal co-retraction e_X being independent of p, q, and s.
(b) Assertions (5.1)-(5.11) remain valid if $\mathbb{R}^n$ is replaced by X.

Proof. Suppose that $X = \mathbb{R}^n_+$. Then (i) and (iii) are proved in [**2**]. Since (i) implies $W_p^s(\mathbb{R}^n_+, E) \doteq r_{\mathbb{R}^n_+} W_p^s(\mathbb{R}^n, E)$, where the latter is given its quotient space topology, assertions (5.4) and (5.9) remain true if $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$. Thus, (ii) and (iv) follow by means of [**8**, Lem.2.3.1, Prop.2.3.2] and the interpolation formulas (5.6) and (5.10). Therefore, (a) holds and also entails (b).
Suppose now that $X = \Omega$. Then we can refer to [**11**, Thm.4.1, Prop.4.2] for (i)-(iii). Again, (iv) follows by interpolation whereas (b) follows from (a). □

Remarks 6.13. (a) Note that (i)-(iii) of the above proposition and assertions (5.1)-(5.8) with $\mathbb{R}^n$ replaced by $X \in \{\mathbb{R}^n_+, \Omega\}$ are true for arbitrary Banach spaces. These are consequences of [**2**] and [**11**, Thm.4.1, Prop.4.2].
(b) Proposition 6.12 remains valid for an unbounded domain Ω provided $\partial\Omega$ is smooth and compact.

Remark 6.14. It is shown in [**53**, Thm.4] that (for an arbitrary Banach space E) the trace operator γ_0, defined by

$$\gamma_0 u(x') := u(x', 0) \ , \quad x' \in \mathbb{R}^{n-1} \ , \quad u \in \mathcal{S}(\mathbb{R}^n, E) \ ,$$

can be extended to

$$\gamma_0 \in \mathcal{L}\big(H_p^s(\mathbb{R}^n, E), B_{p,p}^{s-1/p}(\mathbb{R}^{n-1}, E)\big) \ , \quad 1 < p < \infty \ , \quad s > 1/p \ .$$

Thus, E being a UMD space, we have for $k \in \mathbb{N}$

$$\partial_{e_n}^k := \gamma_0 \partial_n^k e_{\mathbb{R}^n_+} \in \mathcal{L}\big(H_p^s(\mathbb{R}^n_+, E), B_{p,p}^{s-k-1/p}(\mathbb{R}^{n-1}, E)\big) \ , \quad 1 < p < \infty \ , \quad s > k + 1/p \ ,$$

where $e_{\mathbb{R}^n_+}$ denotes a co-retraction of $r_{\mathbb{R}^n_+}$ according to Proposition 6.12.

Before constructing a suitable co-retraction for $\partial_{e_n}^k$, let us introduce the *Poisson semigroup* $\mathcal{P} := \{P(t)\ ;\ t \geq 0\}$ given by $P(t)u := p_t * u$, where $p_0 := \delta$ and

$$p_t(x) := c_n t\big(|x|^2 + t^2\big)^{-(n+1)/2} , \quad x \in \mathbb{R}^n , \quad t > 0 ,$$

with $c_n > 0$ chosen so that $\|p_1\|_{L_1(\mathbb{R}^n)} = 1$. Observe that $p_t \in L_1(\mathbb{R}^n)$ for $t > 0$ and that

$$\widehat{p}_t(\xi) = e^{-t|\xi|} , \quad \xi \in \mathbb{R}^n , \quad t > 0 .$$

Proposition 6.15. *Let $1 < p < \infty$. Then $\mathcal{P}$ is an analytic semigroup of contractions on $L_p(\mathbb{R}^n, E)$. If Λ denotes the infinitesimal generator of $\mathcal{P}$, then $D(\Lambda^m) \doteq H_p^m(\mathbb{R}^n, E)$ and*

$$\Lambda^m u = (-1)^m \mathcal{F}^{-1} |\xi|^m \mathcal{F} u , \quad u \in H_p^m(\mathbb{R}^n, E) ,$$

for each $m \in \mathbb{N}$.

Proof. One proves the assertion along the lines of [**62**, Lem.2.5.3, Rem. 2.5.3/1,2]. In order to characterize the domain of Λ^m, use the Mikhlin multiplier theorem of [**71**]. □

Theorem 6.16. *Let $m \in \mathbb{N}$. Then there exists*

$$Q_m \in \mathcal{L}\Big(\prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\mathbb{R}^{n-1}, E), H_p^s(\mathbb{R}_+^n, E)\Big) , \quad s \in \mathbb{R} , \quad 1 < p < \infty ,$$

such that for each $k \in \{0, \dots, m\}$ with $s > k + 1/p$

$$\partial_{e_n}^k Q_m(u^0, \dots, u^m) = u^k , \quad (u^0, \dots, u^m) \in \prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\mathbb{R}^{n-1}, E) .$$

Proof. (a) Fix $\lambda > 0$ and $j \in \mathbb{N}$. First suppose that $n = 1$. Then $\mathbb{R}^{n-1} = \{0\}$ and hence $S(\mathbb{R}^{n-1}, E) = E$ for $S \in \{W_p^s, H_p^s, B_{p,q}^s\}$. Define $R_j := R_j(\lambda)$ by

$$R_j u(t) := \frac{t^j}{j!} e^{-\lambda t} u , \quad t \geq 0 , \quad u \in E .$$

Clearly,

$$\|\partial_t^k R_j u\|_{L_p((0,\infty),E)} \leq c_k |u|_E , \quad k \in \mathbb{N} , \quad 1 < p < \infty ,$$

and thus trivially

$$R_j = R_j(\lambda) \in \mathcal{L}\big(B_{p,p}^{s-j-1/p}(\mathbb{R}^{n-1}, E), H_p^s(\mathbb{R}_+^n, E)\big) , \quad s \in \mathbb{R} , \quad 1 < p < \infty ,$$

since $B_{p,p}^{s-j-1/p}(\mathbb{R}^{n-1}, E) = E$ and $W_p^k(\mathbb{R}_+^n, E) \doteq H_p^k(\mathbb{R}_+^n, E) \hookrightarrow H_p^s(\mathbb{R}_+^n, E)$ for $s < k \in \mathbb{N}$. Suppose now that $n \geq 2$. Denote by $\mathcal{F}_{n-1}$ the Fourier transform on $\mathbb{R}^{n-1}$ and define $R_j := R_j(\lambda)$ by

$$R_j u(t) := \frac{t^j}{j!} e^{-\lambda t} \mathcal{F}_{n-1}^{-1} e^{-\lambda t|\xi|} \mathcal{F}_{n-1} u , \quad t \geq 0 , \quad u \in \mathcal{S}'(\mathbb{R}^{n-1}, E) .$$

Let $\mathcal{P} = \{P(t)\,;\, t \geq 0\}$ be the Poisson semigroup on $L_p(\mathbb{R}^{n-1}, E)$ with generator Λ. Since $\widehat{p}_{\lambda t}(\xi) = e^{-\lambda t|\xi|}$, $\xi \in \mathbb{R}^{n-1}$, the convolution theorem yields

$$R_j u(t) = \frac{t^j}{j!} e^{-\lambda t} P(\lambda t) u =: Z_j u(t) , \quad t \geq 0 , \quad u \in \mathcal{S}(\mathbb{R}^{n-1}, E) .$$

$\mathcal{S}(\mathbb{R}^{n-1}, E)$ being a dense subset of $L_p(\mathbb{R}^{n-1}, E)$ and $e^{-\lambda t|\cdot|}$ being a Fourier multiplier for $L_p(\mathbb{R}^{n-1}, E)$ (cf. [**71**]), we deduce

$$R_j u(t) = Z_j u(t) , \quad t \geq 0 , \quad u \in L_p(\mathbb{R}^{n-1}, E) . \tag{6.7}$$

On the other hand, (5.10) and Proposition 6.15 guarantee that we may apply [**5**, Prop.B.1] to obtain (see the proof of [**5**, Thm.B.2])

$$Z_j \in \mathcal{L}\big(B^{k-j-1/p}_{p,p}(\mathbb{R}^{n-1},E), H^k_p(\mathbb{R}^n_+,E)\big)\ , \quad k\in\mathbb{N}\ , \quad k\geq j+1\ .$$

Recalling (5.7) and (5.10), complex interpolation and (6.7) result in

$$R_j \in \mathcal{L}\big(B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E), H^s_p(\mathbb{R}^n_+,E)\big)\ , \quad s\in\mathbb{R}\ , \quad s\geq j+1\ .$$

Let $s<j+1$ and choose $k\in\mathbb{N}$ with $s+2k\geq j+1$. Then, for $u\in B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)$,

$$R_j u = \mathcal{F}^{-1}_{n-1}\big(1+|\xi|^2\big)^k\mathcal{F}_{n-1}\frac{t^j}{j!}e^{-\lambda t}\mathcal{F}^{-1}_{n-1}e^{-\lambda t|\xi|}\mathcal{F}_{n-1}\mathcal{F}^{-1}_{n-1}\big(1+|\xi|^2\big)^{-k}\mathcal{F}_{n-1}u\ .$$

The last part is a mapping from $B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)$ onto $B^{s+2k-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)$, and the above considerations entail that the middle part is a mapping from $B^{s+2k-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)$ into $H^{s+2k}_p(\mathbb{R}^n_+,E)$. Finally, one obtains by the first part a mapping from $H^{s+2k}_p(\mathbb{R}^n_+,E)$ into $H^s_p(\mathbb{R}^n_+,E)$. Therefore,

$$R_j=R_j(\lambda) \in \mathcal{L}\big(B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E), H^s_p(\mathbb{R}^n_+,E)\big)\ , \quad s\in\mathbb{R}\ , \quad 1<p<\infty\ .$$

(b) Now let $n\geq 1$ be arbitrary and fix $m\in\mathbb{N}$. Given $\lambda_m>\lambda_{m-1}>\cdots>\lambda_0>0$ one can uniquely determine coefficients $a_{j,r}$ such that

$$\sum_{r=j}^m a_{j,r}=1\ , \quad j=0,\dots,m\ , \tag{6.8}$$

and

$$\sum_{r=j}^m a_{j,r}\lambda_r^{k-j}=0\ , \quad k=1,\dots,m\ , \quad j=0,\dots,k-1\ , \tag{6.9}$$

owing to Vandermonde's determinant. We define

$$Q_m(u^0,\dots,u^m):=\sum_{j=0}^m\sum_{r=j}^m a_{j,r}R_j(\lambda_r)u^j\ , \quad (u^0,\dots,u^m)\in\prod_{j=0}^m B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)\ ,$$

so that in view of (a)

$$Q_m\in\mathcal{L}\Big(\prod_{j=0}^m B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E), H^s_p(\mathbb{R}^n_+,E)\Big)\ , \quad s\in\mathbb{R}\ , \quad 1<p<\infty\ .$$

Assume that $n\geq 2$ and let $u^j\in\mathcal{S}(\mathbb{R}^{n-1},E)$ for each $j\in\{0,\dots,m\}$. Then, for $0\leq k\leq m$,

$$\begin{aligned}\partial^k_{e_n}Q_m(u^0,\dots,u^m)&=\sum_{j=0}^m\sum_{l=0}^k\binom{k}{l}\partial^l_t\frac{t^j}{j!}\Big|_{t=0}\sum_{r=j}^m a_{j,r}\partial^{k-l}_t\big[e^{-\lambda_r t}P(\lambda_r t)u^j\big]\Big|_{t=0}\\ &=\sum_{l=0}^k\binom{k}{l}\sum_{r=l}^m a_{l,r}\lambda_r^{k-l}(-1+\Lambda)^{k-l}u^l\ =\ u^k\end{aligned}$$

according to (6.7)-(6.9). The density of $\mathcal{S}(\mathbb{R}^{n-1},E)$ in $B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)$ entails that for $k\in\{0,\dots,m\}$ with $s>k+1/p$

$$\partial^k_{e_n}Q_m(u^0,\dots,u^m)=u^k\ , \quad (u^0,\dots,u^m)\in\prod_{j=0}^m B^{s-j-1/p}_{p,p}(\mathbb{R}^{n-1},E)\ ,$$

since in this case $\partial_{e_n}^k \in \mathcal{L}\big(H_p^s(\mathbb{R}_+^n, E), B_{p,p}^{s-k-1/p}(\mathbb{R}^{n-1}, E)\big)$. Thus, we have proven the claimed statement if $n \geq 2$. The case $n = 1$ is obtained analogously. □

In order to give a consistent definition of distribution spaces on $\partial\Omega$, we now show that Bessel potential spaces and Besov spaces are invariant under coordinate changes.

If X_1 and X_2 are open subsets of $\mathbb{R}^n$, we denote by $\mathrm{Diff}(X_1, X_2)$ the set of all (smooth) diffeomorphisms from X_1 onto X_2.

Lemma 6.17. *Let X_1, X_2, and X_3 be open subsets of $\mathbb{R}^n$ and let $f \in \mathrm{Diff}(X_1, X_2)$. Then, there exists exactly one $f^* \in \mathcal{L}\big(\mathcal{D}'(X_2, E), \mathcal{D}'(X_1, E)\big)$ such that*

$$f^*u = u \circ f \ , \quad u \in C(X_2, E) \ .$$

It holds

$$\begin{aligned}
(i) &\quad (f^*u)(\varphi) = u\big(|\det \partial f^{-1}|\, \varphi \circ f^{-1}\big) \ , \quad \varphi \in \mathcal{D}(X_1) \ , \quad u \in \mathcal{D}'(X_2, E) \ , \\
(ii) &\quad f^*(au) = f^*a f^*u \ , \quad a \in C^\infty(X_2) \ , \quad u \in \mathcal{D}'(X_2, E) \ , \\
(iii) &\quad (g \circ f)^*u = f^*g^*u \ , \quad u \in \mathcal{D}'(X_3, E) \ , \quad g \in \mathrm{Diff}(X_2, X_3) \ , \\
(iv) &\quad \mathrm{supp}(f^*u) = f^{-1}(\mathrm{supp}\, u) \ , \quad u \in \mathcal{D}'(X_2, E) \ .
\end{aligned}$$

Proof. The proof of [**37**, Thm.6.1.2] can easily be modified for E-valued distributions. □

Proposition 6.18. *Let X_1, $X_2 \subset \mathbb{R}^n$ be open. Suppose that $f \in \mathrm{Diff}(X_1, X_2)$ and that $\chi \in \mathcal{D}(X_2)$. Then, for $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \leq q \leq \infty$,*

$$\big[u \mapsto f^*(\chi u)\big] \in \mathcal{L}\big(H_p^s(\mathbb{R}^n, E)\big) \cap \mathcal{L}\big(B_{p,q}^s(\mathbb{R}^n, E)\big) \ .$$

Proof. Observe that we can extend $f^*(\chi u) = f^*\chi f^*u$ for $u \in \mathcal{D}'(\mathbb{R}^n, E) \subset \mathcal{D}'(X_2, E)$ by zero outside of X_1 to obtain an element of $\mathcal{D}'(\mathbb{R}^n, E)$. Using (5.9) and the density of $\mathcal{S}(\mathbb{R}^n, E)$ in $H_p^m(\mathbb{R}^n, E)$, it is easily seen that

$$\|f^*(\chi u)\|_{H_p^m(\mathbb{R}^n,E)} \leq c\|u\|_{H_p^m(\mathbb{R}^n,E)} \ , \quad u \in H_p^m(\mathbb{R}^n, E) \ , \quad m \in \mathbb{N} \ . \tag{6.10}$$

On the other hand, fix $m \in \dot{\mathbb{N}}$ and let $u \in \mathcal{S}(\mathbb{R}^n, E)$. Then

$$w := (1 - \Delta)^{-m}(\chi u) = \mathcal{F}^{-1}\big(1 + |\xi|^2\big)^{-m}\mathcal{F}(\chi u) \in \mathcal{S}(\mathbb{R}^n, E) \ ,$$

and the chain rule entails

$$(\chi u)(y) = (1 - \Delta)^m w(y) = \sum_{|\alpha| \leq 2m} b_\alpha(y)(f^{-1})^*\partial^\alpha(w \circ f)(y) \ , \quad y \in X_2 \ ,$$

where $b_\alpha \in \mathcal{D}(X_2)$. Let $\varrho \in \mathcal{D}(X_2)$ be with $\varrho = 1$ on a neighbourhood of $\mathrm{supp}\,\chi$ so that

$$f^*(\chi u) = \sum_{|\alpha| \leq 2m} a_\alpha \partial^\alpha\big(f^*(\varrho w)\big) \ ,$$

with $a_\alpha \in \mathcal{D}(X_1)$ being independent of u. Taking into consideration that by [**6**, Thm.2.3] and (5.9)

$$\|\varphi\psi\|_{H_p^{-2m}(\mathbb{R}^n,E)} \leq c_\varphi\|\psi\|_{H_p^{-2m}(\mathbb{R}^n,E)} \ , \quad \psi \in H_p^{-2m}(\mathbb{R}^n, E) \ , \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \ ,$$

we obtain from (5.12) and (6.10)

$$\begin{aligned}\|f^*(\chi u)\|_{H_p^{-2m}(\mathbb{R}^n,E)} &\le c \sum_{|\alpha|\le 2m} \|\partial^\alpha\big(f^*(\varrho w)\big)\|_{H_p^{-2m}(\mathbb{R}^n,E)}\\ &\le c\|f^*(\varrho w)\|_{L_p(\mathbb{R}^n,E)} \le c\|\chi u\|_{H_p^{-2m}(\mathbb{R}^n,E)}\\ &\le c\|u\|_{H_p^{-2m}(\mathbb{R}^n,E)}\ .\end{aligned}$$

$\mathcal{S}(\mathbb{R}^n,E)$ being dense in $H_p^{-2m}(\mathbb{R}^n,E)$, we deduce

$$\big[u\mapsto f^*(\chi u)\big]\in\mathcal{L}\big(H_p^{-2m}(\mathbb{R}^n,E)\big)\cap\mathcal{L}\big(H_p^m(\mathbb{R}^n,E)\big)\ ,\quad m\in\mathbb{N}\ .$$

Invoking (5.10) and (5.11), the assertion is obvious. □

Recall that Ω is a bounded and smooth domain in $\mathbb{R}^n$. Hence we can define for any given $u\in\mathcal{D}'(\partial\Omega,E):=\mathcal{L}\big(C^\infty(\partial\Omega),E\big)$ and any chart (φ,U) of the compact manifold $\partial\Omega$

$$u_\varphi(\psi):=u(\psi\circ\varphi)\ ,\quad \psi\in\mathcal{D}\big(\varphi(U)\big)\ , \tag{6.11}$$

so that $u_\varphi\in\mathcal{D}'\big(\varphi(U),E\big)$. If $S\in\big\{W_p^s,H_p^s,B_{p,q}^s\big\}$ we say that $u\in\mathcal{D}'(\partial\Omega,E)$ belongs to $S(\partial\Omega,E)$ provided (the trivial extension of) χu_φ is an element of $S(\mathbb{R}^{n-1},E)$ for all charts (φ,U) of $\partial\Omega$ and all $\chi\in\mathcal{D}\big(\varphi(U)\big)$. Fix an atlas $\big\{(\varphi_j,U_j)\,;\,1\le j\le N\big\}$ of $\partial\Omega$ and a partition of unity $\{\pi_j\,;\,1\le j\le N\}$ subordinate to $\{U_j\,;\,1\le j\le N\}$. Suppose that $s\in\mathbb{R}$, $1<p<\infty$, and $1\le q\le\infty$. Putting

$$\|u\|_{S(\partial\Omega,E)}:=\sum_{j=1}^N\|(\varphi_j^{-1})^*\pi_j u_{\varphi_j}\|_{S(\mathbb{R}^{n-1},E)}\ ,\quad u\in S(\partial\Omega,E)\ ,$$

it is not difficult to prove on the basis of Proposition 6.18 and Lemma 6.17 that $S(\partial\Omega,E)$ endowed with $\|\cdot\|_{S(\partial\Omega,E)}$ is a well-defined Banach space in the sense that different choices of atlases and partitions of unity lead to equivalent norms. Owing to (5.4) and (5.9), it holds

$$W_p^s(\partial\Omega,E)\doteq B_{p,p}^s(\partial\Omega,E)\ ,\quad s\in\mathbb{R}\setminus\mathbb{Z}\ ,\quad 1<p<\infty\ ,$$

and

$$W_p^m(\partial\Omega,E)\doteq H_p^m(\partial\Omega,E)\ ,\quad m\in\mathbb{Z}\ ,\quad 1<p<\infty\ .$$

In virtue of Proposition 6.18 and Proposition 6.12 we can generalize Remark 6.14 by means of local coordinates to obtain that the trace $\partial_\nu^0:=\big[u\mapsto u\big|_{\partial\Omega}\big]$ induces an operator

$$\partial_\nu^0\in\mathcal{L}\big(H_p^s(\Omega,E),B_{p,p}^{s-1/p}(\partial\Omega,E)\big)\ ,\quad s>1/p\ , \tag{6.12}$$

and that $\big[u\mapsto\partial_\nu^k u=\frac{\partial^k}{\partial\nu^k}u\big]$ induces an operator

$$\partial_\nu^k\in\mathcal{L}\big(H_p^s(\Omega,E),B_{p,p}^{s-k-1/p}(\partial\Omega,E)\big)\ ,\quad s>k+1/p\ ,\quad k\in\dot{\mathbb{N}}\ . \tag{6.13}$$

Observe that (6.12) and (6.13) imply for $1<p<\infty$ and $1\le q\le\infty$

$$\partial_\nu^k\in\mathcal{L}\big(B_{p,q}^s(\Omega,E),B_{p,q}^{s-k-1/p}(\partial\Omega,E)\big)\ ,\quad s>k+1/p\ ,\quad k\in\mathbb{N}\ . \tag{6.14}$$

Indeed, in view of (5.11) and Proposition 6.12(b), it suffices to note that for $s_0\ne s_1$ the embedding

$$\big(B_{p,p}^{s_0}(\partial\Omega,E),B_{p,p}^{s_1}(\partial\Omega,E)\big)_{\theta,q}\hookrightarrow B_{p,q}^{(1-\theta)s_0+\theta s_1}(\partial\Omega,E)\ , \tag{6.15}$$

is a consequence of (5.6) and the fact that

$$\big[u\mapsto(\varphi_j^{-1})^*\pi_j u_{\varphi_j}\big]\in\mathcal{L}\big(B_{p,p}^t(\partial\Omega,E),B_{p,p}^t(\mathbb{R}^{n-1},E)\big)\ ,\quad t\in\mathbb{R}\ ,\quad 1\le j\le N\ .$$

Finally, the analogue of Theorem 6.16 reads as:

Theorem 6.19. *For $m \in \mathbb{N}$ there exists*

$$Q_m \in \mathcal{L}\Big(\prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\partial\Omega, E), H_p^s(\Omega, E)\Big) , \quad s \in \mathbb{R} , \quad 1 < p < \infty ,$$

such that for each $k \in \{0, \ldots, m\}$ with $s > k + 1/p$

$$\partial_\nu^k Q_m(u^0, \ldots, u^m) = u^k , \quad (u^0, \ldots, u^m) \in \prod_{j=0}^{m} B_{p,p}^{s-j-1/p}(\partial\Omega, E) .$$

6.3. General Remarks on Interpolation

We give here a brief introduction to the complex and real interpolation method. For more detailed information we refer to [**35**], [**62**], or [**8**, §I.2].

Let $X_j := \big(X_j, \|\cdot\|_{X_j}\big)$ be $\mathbb{C}$-Banach spaces for $j = 0, 1$. (X_0, X_1) is said to be an *interpolation couple*, if there exists a locally convex space Z with $X_j \hookrightarrow Z$ for $j = 0, 1$. In this case we equip the vector space $X_0 + X_1$ with the norm

$$\|x\|_{X_0+X_1} := \inf\big\{\|x_0\|_{X_0} + \|x_1\|_{X_1} \,;\, x = x_0 + x_1 \in X_0 + X_1\big\} ,$$

so that $X_0 + X_1$ is a well-defined Banach space.
Denote by S the open strip $\big\{z \in \mathbb{C};\, 0 < \operatorname{Re} z < 1\big\}$ and define $\mathcal{F}(X_0, X_1)$ as the set of all $f \in BC(\bar{S}, X_0 + X_1)$ (that is, f is a bounded and continuous function from $\bar{S}$ into $X_0 + X_1$) such that $f|_S$ is holomorphic and

$$\big[t \mapsto f(j + it)\big] \in C_0(\mathbb{R}, X_j) , \quad j = 0, 1 ,$$

where C_0 is the space of all continuous functions vanishing at infinity. Then $\mathcal{F}(X_0, X_1)$ is a Banach space with the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max_{j=0,1} \sup_{t\in\mathbb{R}} \|f(j + it)\|_{X_j} .$$

Given $\theta \in (0, 1)$, define the Banach space $[X_0, X_1]_\theta$ by

$$[X_0, X_1]_\theta := \Big(\big\{x \in X_0 + X_1 \,;\, f(\theta) = x \text{ for some } f \in \mathcal{F}(X_0, X_1)\big\} , \|\cdot\|_{[X_0,X_1]_\theta}\Big) ,$$

where

$$\|x\|_{[X_0,X_1]_\theta} := \inf\big\{\|f\|_{\mathcal{F}(X_0,X_1)} \,;\, f(\theta) = x\big\} .$$

For convenience, put $X_\theta := [X_0, X_1]_\theta$, $0 < \theta < 1$, and $[X_0, X_1]_j := X_j$, $j = 0, 1$. Then the *complex interpolation functor* $[\cdot, \cdot]_\theta$ is *exact of exponent* θ, that is, given any other interpolation couple (Y_0, Y_1) and $T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$, it holds

$$\|T\|_{\mathcal{L}([X_0,X_1]_\theta\,,\,[Y_0,Y_1]_\theta)} \le \|T\|^{1-\theta}_{\mathcal{L}(X_0,Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1,Y_1)} , \quad 0 < \theta < 1 . \tag{6.16}$$

Now, if (X_0, X_1) is an interpolation couple over the reals, we put

$$[X_0, X_1]_\theta := [(X_0)_\mathbb{C}, (X_1)_\mathbb{C}]_\theta \cap (X_0 + X_1) , \quad 0 \le \theta \le 1 ,$$

where $(X_j)_\mathbb{C}$ denotes the complexification of X_j. Then $[\cdot, \cdot]_\theta$ is an exact interpolation functor of exponent θ in this case as well.

For the remainder of this section, suppose that $X_1 \hookrightarrow X_0$.

Remarks 6.20. (a) If $f \in \mathcal{F}(X_0, X_1)$ and $\theta \in (0,1)$, then

$$f(\theta + iy) \in X_\theta \quad \text{with} \quad \|f(\theta+iy)\|_{X_\theta} \le \|f\|_{\mathcal{F}(X_0,X_1)}$$

for all $y \in \mathbb{R}$. This follows from $F \in \mathcal{F}(X_0, X_1)$ where $F(z) := f(z+iy)$ for $z \in \bar{S}$.
(b) The embeddings $X_\beta \hookrightarrow X_\alpha$ for $0 \le \alpha \le \beta \le 1$ are valid, and the norms of this injections can be estimated by a constant independent of α and β. Moreover, for $0 < \theta < 1$ there exists $c(\theta) > 0$ with

$$\|x\|_{X_\theta} \le c(\theta) \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta} \ , \quad x \in X_1 \ .$$

These are consequences of (the proofs of) [**35**, Thm.4.2.1] or [**62**, Thm.1.9.3].
(c) Provided $0 \le \theta_0, \theta_1 \le 1$ and $0 < \eta < 1$, it holds

$$\big[X_{\theta_0}, X_{\theta_1}\big]_\eta \doteq X_{(1-\eta)\theta_0 + \eta\theta_1} \ . \tag{6.17}$$

Note that X_1 need not be dense in X_0. For a proof of this version of the reiteration theorem we refer to [**33**, Lem.1.21].

Lemma 6.21. *Let $f \in \mathcal{F}(X_0, X_1)$ and put $S_\vartheta := \{z \in \mathbb{C};\ \vartheta < \operatorname{Re} z < 1\}$ for $\vartheta \in (0,1)$. Then $f|_{\bar{S}_\vartheta} : \bar{S}_\vartheta \to X_\theta$ is continuous and bounded whereas $f|_{S_\vartheta}$ is holomorphic provided that $0 < \theta < \vartheta < 1$.*

Proof. Let $0 < \theta < \vartheta < 1$. Due to Remarks 6.20 we have $[X_0, X_\vartheta]_{\theta/\vartheta} \doteq X_\theta$ and, in addition,

$$\|f(z)\|_{X_\vartheta} \le c\|f\|_{\mathcal{F}(X_0,X_1)} \ , \quad z \in \bar{S}_\vartheta \ .$$

Therefore,

$$\|f(z) - f(z')\|_{X_\theta} \le c\|f(z) - f(z')\|_{[X_0,X_\vartheta]_{\theta/\vartheta}} \le c\|f(z) - f(z')\|_{X_0}^{1-\theta/\vartheta} \ , \quad z, z' \in \bar{S}_\vartheta \ .$$

This implies that $f|_{\bar{S}_\vartheta} : \bar{S}_\vartheta \to X_\theta$ is continuous and bounded.
Fix $z_0 \in S_\vartheta$, choose $r > 0$ with $\bar{\mathbb{B}}_\mathbb{C}(z_0, r) \subset S_\vartheta$, and let $z \in \mathbb{B}_\mathbb{C}(z_0, r)$. Since $f : S \to X_0$ is holomorphic,

$$f(z) = \frac{1}{2\pi i} \sum_{k=0}^\infty \int_{\partial \mathbb{B}_\mathbb{C}(z_0,r)} \frac{f(\xi)}{(\xi - z_0)^{k+1}}\, d\xi\, (z - z_0)^k \quad \text{in} \quad X_0 \ .$$

Obviously, this power series converges absolutely in X_θ due to $f \in BC(\bar{S}_\vartheta, X_\theta)$. ☐

Let us briefly introduce also the *real interpolation functor* $(\cdot,\cdot)_{\theta,q}$ being defined as follows. Suppose that (X_0, X_1) is a (not necessarily continuously injected) interpolation couple. Put

$$J(t,x) := \max\big\{\|x\|_{X_0}\, ,\, t\|x\|_{X_1}\big\} \ , \quad t > 0 \ , \quad x \in X_0 \cap X_1 \ .$$

Let $\theta \in (0,1)$ and $1 \le q \le \infty$. Then $(X_0, X_1)_{\theta,q}$ is defined as the set of all $x \in X_0 + X_1$ having a representation of the form

$$x = \int_0^\infty v(t)\, \frac{dt}{t} \quad \text{in} \quad X_0 + X_1 \ , \tag{6.18}$$

where $v : \dot{\mathbb{R}}^+ \to X_0 \cap X_1$ is measurable with respect to dt/t and

$$\|t^{-\theta} J\big(t, v(t)\big)\|_{L_q(\dot{\mathbb{R}}^+, \frac{dt}{t})} < \infty \ . \tag{6.19}$$

Equipped with the norm

$$\|x\|_{(X_0,X_1)_{\theta,q}} := \inf \|t^{-\theta} J\big(t, v(t)\big)\|_{L_q(\dot{\mathbb{R}}^+, \frac{dt}{t})} \ ,$$

where the infimum is taken over all v satisfying (6.18) and (6.19), $(X_0, X_1)_{\theta,q}$ is a Banach space. Then the real interpolation functor $(\cdot,\cdot)_{\theta,q}$ is exact of exponent θ, that is, (6.16) is valid if $[\cdot,\cdot]_\theta$ is replaced by $(\cdot,\cdot)_{\theta,q}$.
A connection between the complex and the real interpolation method is given by

$$\big([X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1}\big)_{\eta,q} \doteq (X_0, X_1)_{(1-\eta)\theta_0+\eta\theta_1,q} \ , \tag{6.20}$$

where $\theta_0, \theta_1, \eta \in (0,1)$ with $\theta_0 \neq \theta_1$ and $1 \le q \le \infty$. Furthermore, for $1 \le q \le \infty$ and $0 < \zeta < \eta < \xi < 1$, one has the injections

$$(X_0, X_1)_{\xi,q} \hookrightarrow [X_0, X_1]_\eta \hookrightarrow (X_0, X_1)_{\zeta,q} \ . \tag{6.21}$$

6.4. Interpolation with Boundary Conditions

Based on the previous results, we can prove now the desired interpolation formula (6.1). Throughout this section we assume that E is a Hilbert space, that $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, and that $1 < p < \infty$.

For any closed subset A of $\mathbb{R}^n$ we put

$$H^s_{p,A}(\mathbb{R}^n, E) := \big\{u \in H^s_p(\mathbb{R}^n, E)\,;\, \operatorname{supp} u \subset A\big\} \ ,$$

which is then a closed subset of $H^s_p(\mathbb{R}^n, E)$. Moreover, if $X \subset \mathbb{R}^n$ is open, we set

$$\widetilde{H}^s_p(X, E) := r_X H^s_{p,\bar{X}}(\mathbb{R}^n, E) \ .$$

Observe that $H^s_{p,A}(\mathbb{R}^n, E) = \{0\}$, $s \ge 0$, provided A is a smooth and closed submanifold of $\mathbb{R}^n$ of dimension less than n. Indeed, due to Proposition 6.18 we may assume that $A \subset \mathbb{R}^k$, where $k < n$, so that the claim is evident. Therefore, given $X \in \{\mathbb{R}^n_+, \Omega\}$ and $s \ge 0$, we find for each $u \in \widetilde{H}^s_p(X, E)$ a uniquely determined $\sigma(u) \in H^s_{p,\bar{X}}(\mathbb{R}^n, E)$ with $r_X\sigma(u) = u$. Defining

$$\|u\|_{\widetilde{H}^s_p(X,E)} := \|\sigma(u)\|_{H^s_p(\mathbb{R}^n,E)} \ , \tag{6.22}$$

it follows that $\widetilde{H}^s_p(X, E)$ is a Banach space and that

$$r_X \in \mathcal{L}\big(H^s_{p,\bar{X}}(\mathbb{R}^n, E), \widetilde{H}^s_p(X, E)\big) \ , \quad s \ge 0 \ , \tag{6.23}$$

is a retraction with co-retraction

$$\sigma \in \mathcal{L}\big(\widetilde{H}^s_p(X, E), H^s_{p,\bar{X}}(\mathbb{R}^n, E)\big) \ , \quad s \ge 0 \ . \tag{6.24}$$

In particular, we have

$$\widetilde{H}^s_p(X, E) \hookrightarrow H^s_p(X, E) \ , \quad s \ge 0 \ . \tag{6.25}$$

Concerning the trace operator γ_0, recall Remark 6.14.

Lemma 6.22. *Let $m \in \mathbb{N}$ and suppose that $m + 1/p < s < m + 1 + 1/p$. Then, for any $v \in H^s_p(\mathbb{R}^n, E)$ with $\gamma_0\partial_n^k v = 0$, $0 \le k \le m$, it holds*

$$\partial^\beta \chi_{\mathbb{R}^n_+} v = \chi_{\mathbb{R}^n_+}\partial^\beta v \ , \quad \beta \in \mathbb{N}^n \quad \text{with} \quad |\beta| \le m+1 \ .$$

Proof. Fix any $\beta = (\beta', \beta_n) \in \mathbb{N}^{n-1} \times \dot{\mathbb{N}}$ with $|\beta| \le m+1$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Let $w \in \mathcal{S}(\mathbb{R}^n, E)$ so that

$$\chi_{\mathbb{R}^n_+} w \in L_1(\mathbb{R}^n, E) \cap C^\infty(\mathbb{R}^n \setminus \partial\mathbb{R}^n_+, E) \ .$$

Integration by parts yields

$$\begin{aligned}\big(\partial^\beta \chi_{\mathbb{R}^n_+} w\big)(\varphi) &= \big(\chi_{\mathbb{R}^n_+}\partial^\beta w\big)(\varphi) + \sum_{j=0}^{\beta_n-1}(-1)^j \int_{\mathbb{R}^{n-1}} \partial_{x'}^{\beta'}\partial_n^{\beta_n-j-1} w(x',0)\partial_n^j\varphi(x',0)\, dx' \\ &= \big(\chi_{\mathbb{R}^n_+}\partial^\beta w\big)(\varphi) + \sum_{j=0}^{\beta_n-1}(-1)^j \big(\partial_{x'}^{\beta'}\gamma_0\partial_n^{\beta_n-j-1} w\big)\big(\gamma_0\partial_n^j\varphi\big) \ .\end{aligned} \tag{6.26}$$

For $v \in H^s_p(\mathbb{R}^n, E)$ with $\gamma_0\partial_n^k v = 0$, $0 \le k \le m$, choose $w_l \in \mathcal{S}(\mathbb{R}^n, E)$ with $w_l \to v$ in $H^s_p(\mathbb{R}^n, E)$. Clearly,

$$\chi_{\mathbb{R}^n_+} w_l \to \chi_{\mathbb{R}^n_+} v \quad \text{in} \quad L_p(\mathbb{R}^n, E) \hookrightarrow \mathcal{D}'(\mathbb{R}^n, E) \ . \tag{6.27}$$

Since $|\beta| \le m+1$, Theorem 6.11 yields

$$\chi_{\mathbb{R}^n_+}\partial^\beta w_l \to \chi_{\mathbb{R}^n_+}\partial^\beta v \quad \text{in} \quad H_p^{s-m-1}(\mathbb{R}^n, E) \hookrightarrow \mathcal{D}'(\mathbb{R}^n, E) \ . \tag{6.28}$$

On the other hand, Remark 6.14 entails that for $0 \le j \le \beta_n - 1$

$$\partial_{x'}^{\beta'}\gamma_0\partial_n^{\beta_n-j-1} w_l \to \partial_{x'}^{\beta'}\gamma_0\partial_n^{\beta_n-j-1} v = 0 \quad \text{in} \quad B^{s-m-1/p}_{p,p}(\mathbb{R}^{n-1}, E) \hookrightarrow \mathcal{D}'(\mathbb{R}^{n-1}, E) \ . \tag{6.29}$$

Consequently, if we replace in (6.26) w by w_l, we deduce from (6.27)-(6.29)

$$\big(\partial^\beta \chi_{\mathbb{R}^n_+} v\big)(\varphi) = \big(\chi_{\mathbb{R}^n_+}\partial^\beta v\big)(\varphi) \ , \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \ ,$$

for all $\beta = (\beta', \beta_n) \in \mathbb{N}^{n-1} \times \dot{\mathbb{N}}$ with $|\beta| \le m+1$. Obviously, this formula is also true if $\beta = (\beta', 0) \in \mathbb{N}^{n-1} \times \{0\}$ with $|\beta'| \le m+1$. □

This auxiliary result enables us to give a precise characterization of the spaces $\widetilde{H}^s_p(\Omega, E)$.

Proposition 6.23. *It holds*

(i) $\widetilde{H}^s_p(\Omega, E) \doteq H^s_p(\Omega, E)$ *for* $0 \le s < 1/p$,

(ii) $\widetilde{H}^s_p(\Omega, E) \doteq \big\{u \in H^s_p(\Omega, E)\,;\, \partial_\nu^k u = 0\,,\, 0 \le k \le m\big\}$ *for* $m + 1/p < s < m + 1 + 1/p$.

Proof. By means of local coordinates we may replace Ω by $\mathbb{R}^n_+$ (see Proposition 6.18).
(i) Let $0 \le s < 1/p$. Recall that $\widetilde{H}^s_p(\mathbb{R}^n_+, E) \hookrightarrow H^s_p(\mathbb{R}^n_+, E)$ by (6.25). For given $u \in H^s_p(\mathbb{R}^n_+, E)$ choose any $\tilde{u} \in H^s_p(\mathbb{R}^n, E)$ such that $r_{\mathbb{R}^n_+}\tilde{u} = u$. Theorem 6.11 entails then $\chi_{\mathbb{R}^n_+}\tilde{u} \in H^s_{p,\bar{\mathbb{R}}^n_+}(\mathbb{R}^n, E)$ and hence $u = r_{\mathbb{R}^n_+}\chi_{\mathbb{R}^n_+}\tilde{u} \in \widetilde{H}^s_p(\mathbb{R}^n_+, E)$. According to (6.22), we have

$$\|u\|_{\widetilde{H}^s_p(\mathbb{R}^n_+, E)} = \|\chi_{\mathbb{R}^n_+}\tilde{u}\|_{H^s_p(\mathbb{R}^n, E)} \le c\|\tilde{u}\|_{H^s_p(\mathbb{R}^n, E)} \ .$$

Since this holds for every $\tilde{u} \in H^s_p(\mathbb{R}^n, E)$ with $r_{\mathbb{R}^n_+}\tilde{u} = u$, statement (i) is obvious.
(ii) Suppose that $m + 1/p < s < m + 1 + 1/p$. For $u \in \widetilde{H}^s_p(\mathbb{R}^n_+, E)$ choose $\tilde{u} \in H^s_{p,\bar{\mathbb{R}}^n_+}(\mathbb{R}^n, E)$ with $r_{\mathbb{R}^n_+}\tilde{u} = u$. Denoting for $a \in \mathbb{R}^n$ by $\tau_a w$ the right translation of $w \in \mathcal{D}'(\mathbb{R}^n, E)$, that is,

$$(\tau_a w)(\varphi) := w\big(\varphi(\cdot + a)\big) \ , \quad \varphi \in \mathcal{D}(\mathbb{R}^n) \ ,$$

one easily proves on the basis of (5.8)-(5.10) that

$$\tau_{\lambda e_n}\tilde{u} \to \tilde{u} \quad \text{in} \quad H^s_p(\mathbb{R}^n, E) \quad \text{as} \quad \lambda \to 0+ \ .$$

Since $\mathrm{supp}(\tau_{\lambda e_n}\tilde{u}) \subset \mathbb{R}^n_+$ for each $\lambda > 0$, we conclude

$$\partial^k_{e_n} u = \partial^k_{e_n} r_{\mathbb{R}^n_+}\tilde{u} = 0\ , \quad 0 \le k \le m\ ,$$

where $\partial^k_{e_n}$ is as in Remark 6.14. Conversely, choose $u \in H^s_p(\mathbb{R}^n_+, E)$ with $\partial^k_{e_n} u = 0$ for $0 \le k \le m$. Hence $\tilde{u} := e_{\mathbb{R}^n_+} u \in H^s_p(\mathbb{R}^n, E)$ and $\gamma_0 \partial^k_n \tilde{u} = 0$, $0 \le k \le m$, so that by Lemma 6.22 and Theorem 6.11

$$\partial^\beta \chi_{\mathbb{R}^n_+}\tilde{u} = \chi_{\mathbb{R}^n_+}\partial^\beta \tilde{u} \in H^{s-m-1}_p(\mathbb{R}^n, E)\ , \quad |\beta| \le m+1\ ,$$

and thus $\chi_{\mathbb{R}^n_+}\tilde{u} \in H^s_{p,\bar{\mathbb{R}}^n_+}(\mathbb{R}^n, E)$. Therefore, $u = r_{\mathbb{R}^n_+}\chi_{\mathbb{R}^n_+}\tilde{u} \in \widetilde{H}^s_p(\mathbb{R}^n_+, E)$ and

$$\begin{aligned}\|u\|_{\widetilde{H}^s_p(\mathbb{R}^n_+,E)} = \|\chi_{\mathbb{R}^n_+}\tilde{u}\|_{H^s_p(\mathbb{R}^n,E)} &\le c \sum_{|\beta|\le m+1} \|\partial^\beta \chi_{\mathbb{R}^n_+}\tilde{u}\|_{H^{s-m-1}_p(\mathbb{R}^n,E)} \\ &\le c \sum_{|\beta|\le m+1} \|\partial^\beta \tilde{u}\|_{H^{s-m-1}_p(\mathbb{R}^n,E)} \le c\|\tilde{u}\|_{H^s_p(\mathbb{R}^n,E)} \le c\|u\|_{H^s_p(\mathbb{R}^n_+,E)}\ .\end{aligned}$$

□

In the sequel, for given $s_0, s_1 \in \mathbb{R}$ we put

$$s_\theta := (1-\theta)s_0 + \theta s_1\ , \quad 0 < \theta < 1\ .$$

Lemma 6.24. *Let $s_1 > s_0 \ge 0$ and $0 < \theta < 1$. Then*

$$\big[H^{s_0}_{p,\bar{\Omega}}(\mathbb{R}^n, E), H^{s_1}_{p,\bar{\Omega}}(\mathbb{R}^n, E)\big]_\theta \doteq H^{s_\theta}_{p,\bar{\Omega}}(\mathbb{R}^n, E)\ .$$

PROOF. The embedding from the left to the right is implied by (5.10). Conversely, let $u \in H^{s_\theta}_{p,\bar{\Omega}}(\mathbb{R}^n, E) \hookrightarrow H^{s_\theta}_p(\mathbb{R}^n, E)$ and fix $f \in \mathcal{F}\big(H^{s_0}_p(\mathbb{R}^n, E), H^{s_1}_p(\mathbb{R}^n, E)\big)$ with $f(\theta) = u$. Denoting by $r_c \in \mathcal{L}\big(H^s_p(\mathbb{R}^n, E), H^s_p(\bar{\Omega}^c, E)\big)$, $s \in \mathbb{R}$, the restriction operator to $\bar{\Omega}^c := \mathbb{R}^n \setminus \bar{\Omega}$ and by e_c a corresponding co-retraction (see Remarks 6.13(b)), the definition of $F(z) := f(z) - e_c r_c f(z)$, $z \in \bar{S}$, yields $F \in \mathcal{F}\big(H^{s_0}_{p,\bar{\Omega}}(\mathbb{R}^n, E), H^{s_1}_{p,\bar{\Omega}}(\mathbb{R}^n, E)\big)$ with $F(\theta) = u$. □

Corollary 6.25. *Let $s_1 > s_0 \ge 0$ and $0 < \theta < 1$. Then*

$$\big[\widetilde{H}^{s_0}_p(\Omega, E), \widetilde{H}^{s_1}_p(\Omega, E)\big]_\theta \doteq \widetilde{H}^{s_\theta}_p(\Omega, E)\ .$$

PROOF. This follows from (6.23) and (6.24). □

For the sake of readability, we abbreviate in the following proofs any E-valued distribution space $S(\Omega, E)$ over Ω simply by S. In this sense, for instance, $\widetilde{H}^s_p$ stands for $\widetilde{H}^s_p(\Omega, E)$.

Proposition 6.26. *Let $s_1 > s_0 \ge 0$ and $0 < \theta < 1$. Then*

$$\big[H^{s_0}_p(\Omega, E), \widetilde{H}^{s_1}_p(\Omega, E)\big]_\theta \doteq \widetilde{H}^{s_\theta}_p(\Omega, E)\ .$$

PROOF. Let $u \in \big[H^{s_0}_p, \widetilde{H}^{s_1}_p\big]_\theta \hookrightarrow H^{s_\theta}_p$. Here, the embedding is valid according to (5.10), Proposition 6.12, and (6.25). Due to Proposition 6.23, the injection from the left to the right in the statement holds provided $s_\theta \in (0, 1/p)$. Suppose that there exists $m \in \mathbb{N}$ such that $m + 1/p < s_\theta < m + 1 + 1/p$. Choose $0 < \mu < \eta < \theta$ with

$$m + 1/p < s_\mu < s_\theta < m + 1 + 1/p\ ,$$

and $f \in \mathcal{F}(H_p^{s_0}, \widetilde{H}_p^{s_1})$ such that $f(\theta) = u$. Invoking Lemma 6.21, (6.12), and (6.13) we see that the restriction to $S_\eta = \{z \in \mathbb{C}\,;\ \eta < \operatorname{Re} z < 1\}$ of the function

$$\big[z \mapsto \partial_\nu^k f(z)\big] \in BC\big(\bar{S}_\eta, B_{p,p}^{s_\mu - k - 1/p}(\partial\Omega, E)\big)$$

is holomorphic if $k \in \{0, \ldots, m\}$. By Proposition 6.23 we obtain $\partial_\nu^k f(1+it) = 0$, $t \in \mathbb{R}$, and thus $\partial_\nu^k f(\theta) = \partial_\nu^k u = 0$, $0 \le k \le m$, due to the three lines theorem. Applying again Proposition 6.23, we deduce $u \in \widetilde{H}_p^{s_\theta}$. Continuity being obvious, we therefore have

$$\big[H_p^{s_0}, \widetilde{H}_p^{s_1}\big]_\theta \hookrightarrow \widetilde{H}_p^{s_\theta}\ , \quad s_\theta \notin \mathbb{N} + 1/p\ . \tag{6.30}$$

If $s_\theta \in \mathbb{N} + 1/p$ choose $\varepsilon > 0$ small with $s_{\theta\pm\varepsilon} \notin \mathbb{N} + 1/p$. Corollary 6.25, (6.17), and (6.30) entail then

$$\big[H_p^{s_0}, \widetilde{H}_p^{s_1}\big]_\theta \doteq \big[[H_p^{s_0}, \widetilde{H}_p^{s_1}]_{\theta-\varepsilon}, [H_p^{s_0}, \widetilde{H}_p^{s_1}]_{\theta+\varepsilon}\big]_{1/2} \hookrightarrow \big[\widetilde{H}_p^{s_{\theta-\varepsilon}}, \widetilde{H}_p^{s_{\theta+\varepsilon}}\big]_{1/2} \doteq \widetilde{H}_p^{s_\theta}\ .$$

On the other hand, the reverse embedding is an immediate consequence of Corollary 6.25 and (6.25). □

Given $m \in \mathbb{N}$ we define

$$H_{p,\mathcal{B}_m}^s(\Omega, E) := \begin{cases} \{u \in H_p^s(\Omega, E)\,;\ \partial_\nu^m u = 0\}\ , & s > m + 1/p\ , \\ H_p^s(\Omega, E)\ , & 0 \le s < m + 1/p\ , \end{cases} \tag{6.31}$$

and $H_{p,\mathcal{B}_m}^s := H_{p,\mathcal{B}_m}^s(\Omega, E)$. Analogously we define $B_{p,q;\mathcal{B}_m}^s := B_{p,q;\mathcal{B}_m}^s(\Omega, E)$ for $s \ge 0$ and $1 \le q \le \infty$. In view of (6.12)-(6.14), these are well-defined Banach spaces.

Lemma 6.27. *For $m \in \mathbb{N}$ and $s_1 > s_0 > m + 1/p$, there exists a projection P from $H_p^{s_0}(\Omega, E)$ onto $H_{p,\mathcal{B}_m}^{s_0}(\Omega, E)$ such that its restriction to $H_p^{s_1}(\Omega, E)$ is a projection onto $H_{p,\mathcal{B}_m}^{s_1}(\Omega, E)$.*

Proof. Let Q_m be as in Theorem 6.19 and put

$$Pu := u - Q_m(0, \ldots, 0, \partial_\nu^m u)\ , \quad u \in H_p^{s_0}\ .$$

□

Lemma 6.28. *Let $m \in \mathbb{N}$, $0 < \theta < 1$ and $s_1 > s_0 > m + 1/p$. Then*

$$\big[H_{p,\mathcal{B}_m}^{s_0}(\Omega, E), H_{p,\mathcal{B}_m}^{s_1}(\Omega, E)\big]_\theta \doteq H_{p,\mathcal{B}_m}^{s_\theta}(\Omega, E)\ ,$$

and

$$\big(H_{p,\mathcal{B}_m}^{s_0}(\Omega, E), H_{p,\mathcal{B}_m}^{s_1}(\Omega, E)\big)_{\theta,q} \doteq B_{p,q;\mathcal{B}_m}^{s_\theta}(\Omega, E)\ , \quad 1 \le q \le \infty\ .$$

Proof. Lemma 6.27 entails that we may apply [**62**, Thm.1.17.1/1]. The statements are then evident from Proposition 6.12, (5.10), and (5.11). □

Before we state the next theorem, let us add the following result (which is actually true for an arbitrary UMD space E).

Proposition 6.29. *Let $s_j \in \mathbb{R}$ with $s_1 > s_0$, $1 \le q < \infty$, and $0 < \theta < 1$. Then*

$$\big[B_{p,q}^{s_0}(\partial\Omega, E), B_{p,q}^{s_1}(\partial\Omega, E)\big]_\theta \doteq B_{p,q}^{s_\theta}(\partial\Omega, E)\ .$$

PROOF. Fix an atlas $\{(\varphi_j, U_j)\,;\, 1 \le j \le N\}$ of $\partial\Omega$ and a partition of unity $\{\pi_j\}$ subordinate to $\{U_j\}$. Defining

$$M_j u := (\varphi_j^{-1})^* \pi_j u_{\varphi_j}\ , \quad 1 \le j \le N\ , \quad u \in \mathcal{D}'(\partial\Omega, E)\ ,$$

where u_{φ_j} is given by (6.11), the embedding from the left to the right is obtained as in (6.15).

Conversely, if $u \in B^{s_\theta}_{p,q}(\partial\Omega, E)$ and $\varepsilon > 0$ we find due to (5.7) for each $j \in \{1, \ldots, N\}$ some $f_j \in \mathcal{F}\big(B^{s_0}_{p,q}(\mathbb{R}^{n-1}, E), B^{s_1}_{p,q}(\mathbb{R}^{n-1}, E)\big)$ such that $f_j(\theta) = M_j u$ and

$$\|f_j\|_{\mathcal{F}(B^{s_0}_{p,q}(\mathbb{R}^{n-1},E), B^{s_1}_{p,q}(\mathbb{R}^{n-1},E))} \le c_0 \|M_j u\|_{B^{s_\theta}_{p,q}(\mathbb{R}^{n-1},E)} + \varepsilon\ ,$$

where $c_0 > 0$ is the norm of the injection from the right to the left in (5.7). Choose $\varrho_j \in \mathcal{D}\big(\varphi_j(U_j)\big)$ with $\varrho_j = 1$ on $\operatorname{supp}\big((\varphi_j^{-1})^*\pi_j\big)$ and put

$$F(z)(\psi) := \sum_{j=1}^N f_j(z)\big(\varrho_j (\varphi_j^{-1})^* \psi\big)\ , \quad \psi \in C^\infty(\partial\Omega)\ , \quad z \in \bar{S}\ .$$

Then $F \in \mathcal{F}\big(B^{s_0}_{p,q}(\partial\Omega, E), B^{s_1}_{p,q}(\partial\Omega, E)\big)$ and $F(\theta) = u$ from which we deduce that u belongs to $\big[B^{s_0}_{p,q}(\partial\Omega, E), B^{s_1}_{p,q}(\partial\Omega, E)\big]_\theta$. The assertion follows from the estimate

$$\begin{aligned}
\|u\|_{[B^{s_0}_{p,q}(\partial\Omega,E), B^{s_1}_{p,q}(\partial\Omega,E)]_\theta} &\le \|F\|_{\mathcal{F}(B^{s_0}_{p,q}(\partial\Omega,E), B^{s_1}_{p,q}(\partial\Omega,E))} \\
&\le c \sum_{j=1}^N \|f_j\|_{\mathcal{F}(B^{s_0}_{p,q}(\mathbb{R}^{n-1},E), B^{s_1}_{p,q}(\mathbb{R}^{n-1},E))} \\
&\le c\big(\|u\|_{B^{s_\theta}_{p,q}(\partial\Omega,E)} + \varepsilon\big)\ .
\end{aligned}$$

□

Now we can establish our main theorem of this chapter.

Theorem 6.30. *Suppose that $m \in \mathbb{N}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $H^s_{p,\mathcal{B}_m}(\Omega, E)$ and $B^s_{p,q;\mathcal{B}_m}(\Omega, E)$ be as in (6.31), where E is a Hilbert space. For $0 < \theta < 1$ and $s_1 > s_0 \ge 0$ put $s_\theta := (1-\theta)s_0 + \theta s_1$. Then, provided $s_1,\ s_\theta \ne m + 1/p$, it holds*

$$\big[H^{s_0}_p(\Omega, E), H^{s_1}_{p,\mathcal{B}_m}(\Omega, E)\big]_\theta \doteq H^{s_\theta}_{p,\mathcal{B}_m}(\Omega, E)$$

and

$$\big(H^{s_0}_p(\Omega, E), H^{s_1}_{p,\mathcal{B}_m}(\Omega, E)\big)_{\theta,q} \doteq B^{s_\theta}_{p,q;\mathcal{B}_m}(\Omega, E)\ .$$

PROOF. (a) Concerning complex interpolation, the embedding from the left to the right is obvious if $s_\theta < m + 1/p$. Suppose that $s_\theta > m + 1/p$ and choose $0 < \mu < \eta < \theta < 1$ with $m + 1/p < s_\mu < s_\theta$. Given $f \in \mathcal{F}(H^{s_0}_p, H^{s_1}_{p,\mathcal{B}_m})$, Lemma 6.21 entails that

$$\big[z \mapsto \partial^m_\nu f(z)\big] \in BC\big(\bar{S}_\eta, B^{s_\mu - m - 1/p}_{p,p}(\partial\Omega, E)\big)$$

is holomorphic on S_η. Since $\partial^m_\nu f(1 + it) = 0\,,\ t \in \mathbb{R}$, the three lines theorem implies $\partial^m_\nu f(\theta) = 0$. Hence, the interpolation space on the left is continuously embedded in the one on the right.

It remains to prove the reverse inclusion for the complex interpolation result. We may concentrate on $s_1 > m + 1/p$, since otherwise, the assertion follows from (5.10) and Proposition 6.12. Let $u \in H^{s_\theta}_{p,\mathcal{B}_m}$. According to (6.12), (6.13), and Proposition 6.29 we can choose for each $\varepsilon > 0$ and $k \in \mathbb{N}$ with $s_\theta > k + 1/p$ some

$$f_k \in \mathcal{F}\big(B^{s_0 - k - 1/p}_{p,p}(\partial\Omega, E), B^{s_1 - k - 1/p}_{p,p}(\partial\Omega, E)\big)$$

such that $f_k(\theta)=\partial_\nu^k u$ and

$$\|f_k\|_{\mathcal{F}(B_{p,p}^{s_0-k-1/p}(\partial\Omega,E),B_{p,p}^{s_1-k-1/p}(\partial\Omega,E))}\le c\|\partial_\nu^k u\|_{B_{p,p}^{s_\theta-k-1/p}(\partial\Omega,E)}+\varepsilon\ .$$

Fix $M\in\dot{\mathbb{N}}$ with $M\ge\max\{m,s_\theta\}$ and put

$$F_k(z):=\begin{cases}f_k(z)\ , & s_\theta>k+1/p\ ,\\ 0\ , & \text{else}\ ,\end{cases}\qquad z\in\bar S\ ,\quad 0\le k\le M\ ,$$

as well as

$$G(z):=Q_M\big(F_0(z),\dots,F_{m-1}(z),0,F_{m+1}(z),\dots,F_M(z)\big)\ ,\quad z\in\bar S\ ,$$

where Q_M is given by Theorem 6.19. Since $\partial_\nu^m G(1+it)=0$, $t\in\mathbb{R}$, we have

$$v:=G(\theta)\in\big[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}\big]_\theta\hookrightarrow H_p^{s_\theta}\ . \tag{6.32}$$

Suppose $s_\theta\notin\mathbb{N}+1/p$. Then, in view of Propositions 6.23 and 6.26 together with Theorem 6.19, it holds

$$u-v\in\widetilde H_p^{s_\theta}\doteq\big[H_p^{s_0},\widetilde H_p^{s_1}\big]_\theta\hookrightarrow\big[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}\big]_\theta\ . \tag{6.33}$$

Whence, taking into account (6.32), $u\in\big[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}\big]_\theta$. Furthermore, we deduce

$$\begin{aligned}\|v\|_{H_p^{s_\theta}}&\le c\|v\|_{[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}]_\theta}\le c\|G\|_{\mathcal{F}(H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1})}\\ &\le c\sum_{\substack{0\le k\le M\\ s_\theta>k+1/p}}\|f_k\|_{\mathcal{F}(B_{p,p}^{s_0-k-1/p}(\partial\Omega,E),B_{p,p}^{s_1-k-1/p}(\partial\Omega,E))}\\ &\le c\sum_{\substack{0\le k\le M\\ s_\theta>k+1/p}}\|\partial_\nu^k u\|_{B_{p,p}^{s_\theta-k-1/p}(\partial\Omega,E)}+c\varepsilon\le c\big(\|u\|_{H_p^{s_\theta}}+\varepsilon\big)\ ,\end{aligned}$$

where we used only references cited above. Also, (6.33) and Proposition 6.23 imply

$$\|u-v\|_{[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}]_\theta}\le c\|u-v\|_{H_p^{s_\theta}}\ ,$$

and thus

$$\|u\|_{[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}]_\theta}\le c\big(\|u\|_{H_p^{s_\theta}}+\varepsilon\big)\ .$$

Since $\varepsilon>0$ being arbitrary, we obtain from what has already been proven

$$\big[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}\big]_\theta\doteq H_{p,\mathcal{B}_m}^{s_\theta}\ ,\quad s_\theta\notin\mathbb{N}+1/p\ . \tag{6.34}$$

Suppose now $s_\theta=k+1/p$ with $k\in\mathbb{N}\setminus\{m\}$ and fix $\varepsilon>0$ small with $s_{\theta\pm\varepsilon}\notin\mathbb{N}+1/p$ and

$$s_{\theta\pm\varepsilon}\ \begin{Bmatrix}>\\<\end{Bmatrix}\ m+1/p\quad\text{for}\quad s_\theta\ \begin{Bmatrix}>\\<\end{Bmatrix}\ m+1/p\ .$$

Observing that by (6.17) and (6.34)

$$\big[H_p^{s_0},H_{p,\mathcal{B}_m}^{s_1}\big]_\theta\doteq\big[H_{p,\mathcal{B}_m}^{s_{\theta-\varepsilon}},H_{p,\mathcal{B}_m}^{s_{\theta+\varepsilon}}\big]_{1/2}\ ,$$

we deduce from Lemma 6.28 the assertion concerning complex interpolation provided $s_\theta>m+1/p$. But the case $s_\theta<m+1/p$ is obvious.

(b) Finally, the real interpolation result follows exactly as in the last step of part (a) by means of reiteration in virtue of (6.20). ☐

Remark 6.31. Observe that one may generalize (some of) the results of Guidetti [**33**] to Hilbert-space-valued Besov spaces using the same arguments as above. More precisely, one obtains

$$\left[B^{s_0}_{p,q}(\Omega, E), B^{s_1}_{p,q;\mathcal{B}_m}(\Omega, E)\right]_\theta \doteq B^{s_\theta}_{p,q;\mathcal{B}_m}(\Omega, E) \ , \quad q < \infty \ ,$$

and

$$\left(B^{s_0}_{p,q_0}(\Omega, E), B^{s_1}_{p,q_1;\mathcal{B}_m}(\Omega, E)\right)_{\theta,q} \doteq B^{s_\theta}_{p,q;\mathcal{B}_m}(\Omega, E) \ ,$$

provided $1 < p < \infty$, $1 \leq q$, q_0, $q_1 \leq \infty$, and $s_1 > s_0 \geq 0$.

7. On Coalescence and Breakage Equations with Diffusion

We now focus our attention on solvability of Problem $(**)$, see page 57. Throughout this chapter we assume that $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, that $1 < p < \infty$, and that $1 \le \mathsf{p} < \infty$.

We define $E := L_2(Y)$ and denote again by $Y = (0, y_0]$ the range of all possible masses. Moreover, for a Banach space X, we put $F[X] := F(\Omega, X)$, where $F(\Omega, X)$ is any space of X-valued functions defined on Ω.

7.1. The Reaction Terms

Let $E_0, \dots, E_m$ be Banach spaces. Then the Banach space $\mathcal{L}(E_1, \dots, E_m; E_0)$ consists of all continuous m-linear maps from $E_1 \times \cdots \times E_m$ into E_0. They are said to be *multiplications* and are sometimes simply denoted by $(e_1, \dots, e_m) \mapsto e_1 \bullet \cdots \bullet e_m$. Given such a multiplication we define $u_1 \bullet \cdots \bullet u_m \in (E_0)^\Omega$ for $u_j \in (E_j)^\Omega$, $1 \le j \le m$, by

$$u_1 \bullet \cdots \bullet u_m(x) := u_1(x) \bullet \cdots \bullet u_m(x) \ , \quad x \in \Omega \ . \tag{7.1}$$

Finally, for any Banach spaces $F_j[E_j]$ of E_j-valued functions defined on Ω, $1 \le j \le m$, we write

$$F_1[E_1] \bullet \cdots \bullet F_m[E_m] \hookrightarrow F_0[E_0] \ ,$$

if the point-wise product (7.1) defines an element of $\mathcal{L}\big(F_1[E_1], \dots, F_m[E_m]; F_0[E_0]\big)$.

Lemma 7.1. *(a) Suppose that* $E_1 \times \cdots \times E_m \to E_0$, $(e_1, \dots, e_m) \mapsto e_1 \bullet \cdots \bullet e_m$ *is a multiplication with* $m \ge 3$. *If*

$$0 < \tau < \min\{r, n/p\} \quad \textit{and} \quad \tau + n/p < 2\sigma \ , \tag{7.2}$$

then

$$BUC^r[E_1] \bullet \cdots \bullet BUC^r[E_{m-2}] \bullet B^\sigma_{p,\mathsf{p}}[E_{m-1}] \bullet B^\sigma_{p,\mathsf{p}}[E_m] \hookrightarrow B^\tau_{p,\mathsf{p}}[E_0] \ .$$

(b) Suppose that $E_1 \times E_2 \to E_0$, $(e_1, e_2) \mapsto e_1 \bullet e_2$ *is a multiplication. If* $0 < \sigma < r$, *then*

$$BUC^r[E_1] \bullet B^\sigma_{p,\mathsf{p}}[E_2] \hookrightarrow B^\sigma_{p,\mathsf{p}}[E_0] \ .$$

Proof. According to Proposition 6.12(a) and Remarks 6.13(a), we may assume that $\Omega = \mathbb{R}^n$. Then the assertions are consequences of (5.5) and [**6**, Thm.4.1, Rem.4.2(b)], if one observes that the results in [**6**] remain valid for arbitrary, not necessarily finite dimensional Banach spaces (see [**9**] and [**12**]). □

The space $C_b^{1-}(E_1, E_0)$ consists of all maps from E_1 into E_0 which are uniformly Lipschitz continuous on bounded subsets of E_1. Endowed with the family of seminorms

$$p_B := \Big[u \mapsto \sup_{e \in B} \|u(e)\|_{E_0} + \sup_{\substack{e, e' \in B \\ e \ne e'}} \frac{\|u(e) - u(e')\|_{E_0}}{\|e - e'\|_{E_1}} \Big] \ ,$$

where B runs through the family of all bounded subsets of E_1, $C_b^{1-}(E_1, E_0)$ is a locally convex space. Then $C^\rho\big(\mathbb{R}^+, C_b^{1-}(E_1, E_0)\big)$ for $\rho \in (0,1)$ is also a locally convex space, where the topology is induced by the family of seminorms

$$u \mapsto \max_{0 \le t \le T} p_B\big(u(t)\big) + \sup_{0 \le s < t \le T} \frac{p_B\big(u(t) - u(s)\big)}{|t - s|^\rho} \ ,$$

with $T>0$ and $B\subset E_1$ bounded.

Let us give a precise description how the right hand side of Problem $(**)$ will be interpreted in the following.
We set $F_{break}:=L_\infty(Y,E)$ and use the notation

$$\gamma(y,y'):=\gamma(y)(y')\ ,\quad \text{a.a. } y,y'\in Y\ ,\quad \gamma\in F_{break}\ .$$

Given $\gamma\in F_{break}$, we define

$$l_b(\gamma)[u](y):=\int_y^{y_0}\gamma(y',y)u(y')\ dy'-u(y)\int_0^y\frac{y'}{y}\gamma(y,y')\ dy'$$

for $u\in E$ and a.a. $y\in Y$. Since Y is a bounded interval, it is easily verified that

$$\big[(\gamma,u)\mapsto l_b(\gamma)[u]\big]\in\mathcal{L}(F_{break},E;E)\ ,$$

and hence, by putting

$$l_b(\gamma)[u](x):=l_b\big(\gamma(x)\big)[u(x)]\ ,\quad x\in\Omega\ ,$$

for $(\gamma,u):\Omega\to F_{break}\times E$, Lemma 7.1(b) yields a multiplication

$$\big[(\gamma,u)\mapsto l_b(\gamma)[u]\big]\in\mathcal{L}\big(BUC^r[F_{break}],B^\sigma_{p,\mathsf{p}}[E];B^\sigma_{p,\mathsf{p}}[E]\big)\ ,$$

provided $0<\sigma<r$. If $\gamma\in C^\rho\big(\mathbb{R}^+,BUC^r[F_{break}]\big)$ with $\rho\in(0,1)$ is fixed, we define

$$l_b(t,x,y,u):=l_b\big(\gamma(t)(x)\big)[u](y)\ ,\quad (t,x)\in\mathbb{R}^+\times\Omega\ ,\quad \text{a.a. } y\in Y\ ,\quad u\in E\ .$$

Denoting then by $L_b(t,\cdot)$ the Nemitskii operator induced by $l_b(t,\cdot,\cdot,\cdot)$, that is,

$$L_b(t,u)(x):=l_b\big(t,x,\cdot,u(x)\big)\ ,\quad (t,x)\in\mathbb{R}^+\times\Omega\ ,\quad u\in E^\Omega\ ,$$

we deduce that

$$\big[t\mapsto L_b(t,\cdot)\big]\in C^\rho\Big(\mathbb{R}^+,C_b^{1-}\big(B^\sigma_{p,\mathsf{p}}[E],B^\sigma_{p,\mathsf{p}}[E]\big)\Big)\ ,\quad 0<\sigma<r\ .$$

Next, let F_{coal} be the closed linear subspace of $L_\infty(Y\times Y)$ consisting of all R satisfying

$$R(y,y')=R(y',y)\ ,\quad \text{a.a. } y,y'\in Y\ .$$

Defining

$$l_c^1(K,P)[u,v](y):=\frac{1}{2}\int_0^y K(y',y-y')P(y',y-y')u(y')v(y-y')\ dy'$$

for $K,P\in F_{coal}$, $u,v\in E$, and a.a. $y\in Y$, we obtain a multiplication

$$\big[(K,P,u,v)\mapsto l_c^1(K,P)[u,v]\big]\in\mathcal{L}(F_{coal},F_{coal},E,E;E)\ .$$

Similarly, the definitions of

$$l_c^2(\beta_c,K,Q)[u,v](y):=\frac{1}{2}\int_y^{y_0}\int_0^{y'}K(y'',y'-y'')Q(y'',y'-y'')\beta_c(y',y)u(y'')v(y'-y'')\ dy''dy'$$

and

$$l_c^3(K,R)[u,v](y):=u(y)\int_0^{y_0-y}K(y,y')R(y,y')v(y')\ dy'\ ,$$

for $\beta_c\in F_{break}$, $K,R,Q\in F_{coal}$, $u,v\in E$, and a.a. $y\in Y$, yield multiplications

$$\big[(\beta_c,K,Q,u,v)\mapsto l_c^2(\beta_c,K,Q)[u,v]\big]\in\mathcal{L}(F_{break},F_{coal},F_{coal},E,E;E)\ ,$$

and

$$\big[(K,R,u,v)\mapsto l_c^3(K,R)[u,v]\big]\in\mathcal{L}(F_{coal},F_{coal},E,E;E)\ .$$

Further, for $\beta_s \in F_{scat} := L_\infty\big((y_0, 2y_0], E\big)$, we write

$$\beta_s(y, y') := \beta_s(y)(y') \ , \quad y \in (y_0, 2y_0] \ , \quad y' \in Y \ .$$

We then put

$$l_s^1(\beta_s, K)[u,v](y) := \frac{1}{2}\int_{y_0}^{2y_0}\int_{y'-y_0}^{y_0} K(y'', y'-y'')\beta_s(y', y)u(y'')v(y'-y'')\ dy''dy' \ ,$$

and

$$l_s^2(K)[u,v](y) := u(y)\int_{y_0-y}^{y_0} K(y, y')v(y')\ dy' \ ,$$

for $\beta_s \in F_{scat}$, $K \in F_{coal}$, $u, v \in E$, and a.a. $y \in Y$. Then, the mappings l_s^1 and l_s^2 have the property

$$\big[(\beta_s, K, u, v) \mapsto l_s^1(\beta_s, K)[u,v]\big] \in \mathcal{L}(F_{scat}, F_{coal}, E, E; E) \ ,$$

and

$$\big[(K, u, v) \mapsto l_s^2(K)[u,v]\big] \in \mathcal{L}(F_{coal}, E, E; E) \ .$$

Suppose now that

$$(\beta_c, \beta_s, K, P, Q) \in C^\rho\big(\mathbb{R}^+, BUC^r[F_{break} \times F_{scat} \times F_{coal} \times F_{coal} \times F_{coal}]\big)$$

is fixed, where $r > 0$ and $\rho \in (0,1)$. We set

$$\begin{aligned} l_c(t,x,y,u,v) &:= l_c^1(t,x,y,u,v) + l_c^2(t,x,y,u,v) - l_c^3(t,x,y,u,v) \\ &:= l_c^1\big(K(t)(x), P(t)(x)\big)[u,v](y) \\ &\quad + l_c^2\big(\beta_c(t)(x), K(t)(x), Q(t)(x)\big)[u,v](y) \\ &\quad - l_c^3\big(K(t)(x), P(t)(x) + Q(t)(x)\big)[u,v](y) \end{aligned}$$

as well as

$$\begin{aligned} l_s(t,x,y,u,v) &:= l_s^1(t,x,y,u,v) - l_s^2(t,x,y,u,v) \\ &:= l_s^1\big(\beta_s(t)(x), K(t)(x)\big)[u,v](y) - l_s^2\big(K(t)(x)\big)[u,v](y) \end{aligned}$$

for $(t,x) \in \mathbb{R}^+ \times \Omega$, $u, v \in E$, and a.a. $y \in Y$. Moreover, we denote by $L_h^j(t,\cdot,\cdot)$ for $(j,h) \in \big\{(1,c),(2,c),(3,c),(1,s),(2,s)\big\}$ the Nemitskii operators being induced by $l_h^j(t,\cdot,\cdot,\cdot,\cdot)$, that is,

$$L_h^j(t,u,v)(x) := l_h^j\big(t,x,\cdot,u(x),v(x)\big) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad u, v \in E^\Omega \ ,$$

and we put

$$L_c(t,u) := L_c^1(t,u,u) + L_c^2(t,u,u) - L_c^3(t,u,u) \ , \quad t \in \mathbb{R}^+ \ , \quad u \in E^\Omega \ ,$$

and

$$L_s(t,u) := L_s^1(t,u,u) - L_s^2(t,u,u) \ , \quad t \in \mathbb{R}^+ \ , \quad u \in E^\Omega \ .$$

Therefore, in virtue of Lemma 7.1, these operators satisfy

$$\big[t \mapsto L_h(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+, C_b^{1-}\big(B_{p,\mathsf{p}}^\sigma[E], B_{p,\mathsf{p}}^\tau[E]\big)\big) \ , \quad h \in \{c, s\} \ ,$$

provided (7.2) holds. Finally, we set

$$L(t,\cdot) := L_b(t,\cdot) + L_c(t,\cdot) + L_s(t,\cdot) \ , \quad t \in \mathbb{R}^+ \ , \tag{7.3}$$

and

$$\mathbb{F} := F_{break} \times F_{break} \times F_{scat} \times F_{coal} \times F_{coal} \times F_{coal} \ . \tag{7.4}$$

We summarize the observations above in the following proposition.

Proposition 7.2. *Assume that* $1 < p < \infty$, $1 \le \mathsf{p} < \infty$, *and let* $0 < \tau < \min\{r, n/p\}$ *with* $\tau + n/p < 2\sigma$. *Suppose that*

$$\Big[t \mapsto \big(\gamma(t), \beta_c(t), \beta_s(t), K(t), P(t), Q(t)\big)\Big] \in C^\rho\big(\mathbb{R}^+, BUC^r[\mathbb{F}]\big)$$

for some $\rho \in (0,1)$. *Then it holds*

$$\big[t \mapsto L(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+, C_b^{1-}\big(B^\sigma_{p,\mathsf{p}}[E], B^\tau_{p,\mathsf{p}}[E]\big)\big) \ .$$

For the sake of readability we will use in the sequel the notation

$$a(t,x;\cdot,\cdot) := a(t)(x)(\cdot,\cdot) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad a \in \big\{\gamma, \beta_c, \beta_s, K, P, Q\big\} \ .$$

Remark 7.3. Of course, since the physical meaning of P and Q is that they represent the probability of coalescence and shattering, respectively, they obey (besides being non-negative)

$$0 \le P(t,x;y,y') + Q(t,x;y,y') \le 1 \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ , \quad \text{a.a. } y,y' \in Y \ .$$

But from a mathematical point of view, this will not be required in the following.

7.2. The Diffusion Semigroup

Given $d \in C(\bar\Omega \times \bar Y)$ and $1 < \sigma < \infty$ we set

$$A_p[d]u := -d\Delta u \ , \quad u \in H^2_{p,\mathcal{B}}[L_\sigma(Y)] := \big\{u \in H^2_p[L_\sigma(Y)] \, ; \, \partial_\nu u = 0\big\} \ . \tag{7.5}$$

Since

$$L_\infty(Y) \times L_\sigma(Y) \to L_\sigma(Y) \ , \quad (\varphi, u) \mapsto \varphi u$$

is a multiplication and since $d(x,\cdot) \in L_\infty(Y)$ for $x \in \bar\Omega$, it follows from (5.9) and Proposition 6.12 that

$$\big[d \mapsto A_p[d]\big] \in \mathcal{L}\big(C(\bar\Omega \times \bar Y), \mathcal{L}\big(H^2_{p,\mathcal{B}}[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)\big) \ . \tag{7.6}$$

In the sequel, if E_0 and E_1 are Banach spaces with $E_1 \overset{d}{\hookrightarrow} E_0$ and if $A : E_1 \to E_0$ is linear, we mean by writing $A \in \mathcal{H}(E_1, E_0)$ that $-A$, considered as a linear operator in E_0 with domain E_1, generates an analytic semigroup on E_0. Observe that $\mathcal{H}(E_1, E_0)$ is an open subset of $\mathcal{L}(E_1, E_0)$.

Theorem 7.4. *Let* $d \in C(\bar\Omega \times \bar Y)$ *be with*

$$d(x,y) > 0 \ , \quad (x,y) \in \bar\Omega \times \bar Y \ , \tag{7.7}$$

and define $A_p[d]$ *by (7.5). Then it holds*

$$A_p[d] \in \mathcal{H}\big(H^2_{p,\mathcal{B}}[L_\sigma(Y)], L_p[L_\sigma(Y)]\big) \ , \quad 1 < p, \sigma < \infty \ .$$

Proof. Put $A := A_p[d]$. Let us then verify the hypotheses of [**23**, Thm.8.2]. First observe that $L_\sigma(Y)$ is a UMD space (see [**8**, III.Thm.4.5.2]). Next, condition (7.7) guarantees that, for each $(x,\xi) \in \bar\Omega \times \mathbb{R}^n$ with $|\xi| = 1$, the spectrum of the principal symbol

$$A_\sharp(x,\xi) = d(x,\cdot)|\xi|^2 = d(x,\cdot) \in \mathcal{L}\big(L_\sigma(Y)\big)$$

is contained in $(0,\infty)$. In particular, $A_\sharp(x,\xi)$ is parameter-elliptic.
Further, we have to check the Lopatinskii-Shapiro Condition of [**23**]. Fix $x_0 \in \partial\Omega$,

$\varphi \in (\pi/2, \pi)$, $\lambda \in \big[|\arg z| \le \varphi\big]$, and $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \neq 0$. We then show that the problem

$$\begin{aligned} \big(\lambda + d(x_0,\cdot)|\xi'|^2\big)v(t) - d(x_0,\cdot)\ddot{v}(t) &= 0 \ , \quad t > 0 \ , \\ \dot{v}(0) &= h \end{aligned} \tag{7.8}$$

has for each $h \in L_\sigma(Y)$ a unique solution v in $C_0\big(\mathbb{R}^+, L_\sigma(Y)\big)$. For, put

$$M(y) := \frac{\lambda}{d(x_0, y)} + |\xi'|^2 \ , \quad y \in Y \ ,$$

so that, in view of (7.7), $M \in L_\infty(Y)$ with $M(y) \notin (-\infty, 0]$ for $y \in Y$. Denote by $\sqrt{M(y)}$, $y \in Y$, the unique square root of $M(y)$ with positive real part. Then there exists $m_0 > 0$ such that

$$\operatorname{Re}\sqrt{M(y)} \ge m_0 \ , \quad y \in Y \ . \tag{7.9}$$

Let $h \in L_\sigma(Y)$ be fixed. Clearly, by rewriting (7.8) as a first order differential equation, we see that its unique solution $v \in C^2\big(\mathbb{R}^+, L_\sigma(Y)\big)$ satisfying $v(0) = v^0 \in L_\sigma(Y)$ is given by

$$v(t; v^0) := \frac{1}{2}\Big(v^0 + \frac{h}{\sqrt{M}}\Big)e^{\sqrt{M}t} + \frac{1}{2}\Big(v^0 - \frac{h}{\sqrt{M}}\Big)e^{-\sqrt{M}t} \ , \quad t \ge 0 \ .$$

Hence, it suffices to prove that there exists a uniquely determined $v^0 \in L_\sigma(Y)$ such that $v(\cdot; v^0)$ vanishes at infinity. Due to (7.9), this is indeed the case with $v^0 := -h/\sqrt{M}$.
We may now apply [**23**, Thm.8.2] in order to deduce that there exists $\mu \ge 0$ such that $\mu + A$ is $\mathcal{R}$-sectorial with spectral angle strictly less than $\pi/2$ in the sense of [**23**, Def.4.1]. In particular, $-(\mu + A)$ is the generator of an analytic semigroup on $L_p[L_\sigma(Y)]$ with domain $H^2_{p,\mathcal{B}}[L_\sigma(Y)]$ (see [**8**]), and whence also $A \in \mathcal{H}\big(H^2_{p,\mathcal{B}}[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)$.

□

Given $d : \mathbb{R}^+ \to C(\bar{\Omega} \times \bar{Y})$ we set $d(t, x, y) := d(t)(x, y)$ for $(t, x, y) \in \mathbb{R}^+ \times \bar{\Omega} \times \bar{Y}$.

Corollary 7.5. *Suppose $\rho \in (0, 1)$ and let $d \in C^\rho\big(\mathbb{R}^+, C(\bar{\Omega} \times \bar{Y})\big)$ be with*

$$d(t, x, y) > 0 \ , \quad (t, x, y) \in \mathbb{R}^+ \times \bar{\Omega} \times \bar{Y} \ .$$

Then it holds for $1 < p, \sigma < \infty$

$$\big[t \mapsto A_p[d(t)]\big] \in C^\rho\Big(\mathbb{R}^+, \mathcal{H}\big(H^2_{p,\mathcal{B}}[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)\Big) \ .$$

Proof. This follows from Theorem 7.4 and (7.6). □

On condition that Corollary 7.5 holds, [**8**, II.Cor.4.4.2] guarantees now the existence of an evolution operator U_{A_p} of $A_p := A_p[d]$ on $L_p[L_\sigma(Y)]$. In the following, we collect some basic properties of U_{A_p}, which will be of importance later in the proof of positivity of solutions.

Lemma 7.6. *Assume that d satisfies the hypotheses of Corollary 7.5 and suppose that $1 < \sigma < \infty$. Then*

$$U_{A_p}\big|_{L_q[L_\sigma(Y)]} = U_{A_q} \ , \quad 1 < p < q < \infty \ .$$

Proof. In this proof we abbreviate any space $S[L_\sigma(Y)]$ of $L_\sigma(Y)$-valued functions defined on Ω simply by S.
(i) Fix $s \ge 0$ arbitrarily and let $\big\{e^{-tA_p(s)} \, ; \, t \ge 0\big\}$ denote the semigroup being generated

by $-A_p(s)$. In view of (5.9) and Proposition 6.12(b) we have $H^2_{q,\mathcal{B}} \hookrightarrow H^2_{p,\mathcal{B}}$, since Ω is bounded, and whence $A_p(s) \supset A_q(s)$. From this it immediately follows

$$\big(\lambda + A_p(s)\big)^{-1}\big|_{L_q} = \big(\lambda + A_q(s)\big)^{-1} , \quad \lambda \in \varrho\big(- A_p(s)\big) \cap \varrho\big(- A_q(s)\big) ,$$

with $\varrho\big(- A_p(s)\big)$ denoting the resolvent set of $-A_p(s)$. Invoking then [**34**, Thm.11.6.6] we deduce

$$e^{-tA_p(s)}u = \lim_{k\to\infty} \Big(\frac{k}{t}\Big)^k \Big(\frac{k}{t} + A_p(s)\Big)^{-k} u = \lim_{k\to\infty} \Big(\frac{k}{t}\Big)^k \Big(\frac{k}{t} + A_q(s)\Big)^{-k} u = e^{-tA_q(s)}u$$

for $u \in L_q \hookrightarrow L_p$ and $t > 0$. Therefore,

$$e^{-tA_p(s)}\big|_{L_q} = e^{-tA_q(s)} , \quad t, s \geq 0 .$$

(ii) Let $T > 0$ be arbitrary and put

$$a_p(t,s) := e^{-(t-s)A_p(s)} , \quad 0 \leq s \leq t \leq T .$$

Hence, by (i) we have $a_p(t,s)\big|_{L_q} = a_q(t,s)$. Moreover, set

$$k_p(t,s) := -\big(A_p(t) - A_p(s)\big)a_p(t,s) , \quad 0 \leq s \leq t \leq T ,$$

so that $k_p(t,s)\big|_{L_q} = k_q(t,s)$. Finally, define

$$\omega_p := \sum_{j=1}^{\infty} \underbrace{k_p * \cdots * k_p}_{j \text{ times}} ,$$

by means of

$$k * h(t,s) := \int_s^t k(t,\tau)h(\tau,s)\, d\tau , \quad 0 \leq s \leq t \leq T .$$

Since [**8**, II.Lem.4.3.1] entails for $\Sigma_T := \big\{(t,s)\, ;\, 0 \leq s < t \leq T\big\}$ that $\omega_p \in C\big(\Sigma_T, \mathcal{L}(L_p)\big)$ and $\omega_p(t,t) = 0$, we obtain from $k_p\big|_{L_q} = k_q$ that

$$\omega_p(t,s)\big|_{L_q} = \omega_q(t,s) , \quad 0 \leq s \leq t \leq T .$$

Because [**8**, II.§4.3, II.§4.4] tells us

$$U_{A_p}(t,s) = a_p(t,s) + a_p * \omega_p(t,s) , \quad 0 \leq s \leq t \leq T ,$$

the assertion is obvious. □

For a vector space X being ordered by a proper cone X^+ (that is, $x \leq y$ iff $y - x \in X^+$ with the convention that $y \geq x$ iff $x \leq y$) and any set M, the vector space X^M is given its point-wise order induced by the cone $\big(X^+\big)^M$. This means that $w \leq v$ for $w, v \in X^M$ iff $w(m) \leq v(m)\, ,\, m \in M$. If X is a locally convex space then X is an ordered locally convex space provided X is an ordered vector space whose positive cone X^+ is closed. In particular, if X is an ordered Banach space then $L_\sigma(Y)$ and $L_p[X]$ are ordered Banach spaces (with point-wise order a.e.) with cones $L^+_\sigma(Y)$ and $L^+_p[X]$, respectively. Given

$$S \in \big\{BUC^\mu[X], H^\mu_p[X], W^\mu_p[X], B^\mu_{p,q}[X]\, ;\, \mu > 0\big\}$$

the order of S is defined by the cone $S^+ := S \cap L^+_p[X]$.

We denote by $C_c(Y)$ the space of all continuous functions on Y with compact supports and by $C^+_c(Y)$ its positive cone.

For a definition of the tensor product $E_1 \otimes E_2$ we refer to [**8**].

Lemma 7.7. *If* $1 < p, \sigma < \infty$ *then* $BUC^\infty(\Omega)^+ \otimes C_c^+(Y)$ *is dense in* $L_p^+[L_\sigma(Y)]$ *and* $BUC^\infty(\Omega) \otimes C_c(Y)$ *is dense in* $W_p^2[L_\sigma(Y)]$.

Proof. Clearly, the trivial extension $\tilde{v}$ of $v \in L_p^+[L_\sigma(Y)]$ belongs to $L_p^+\big(\mathbb{R}^n, L_\sigma(Y)\big)$. We may now approximate $\tilde{v}$ by functions of the tensor product $\mathcal{D}^+(\mathbb{R}^n) \otimes C_c^+(Y)$ similarly as in [**10**, Lem.6.1]. Thus $BUC^\infty(\Omega)^+ \otimes C_c^+(Y)$ is indeed dense in $L_p^+[L_\sigma(Y)]$. Extending elements of $W_p^2[L_\sigma(Y)]$ by means of a co-retraction according to Proposition 6.12, the second assertion is obtained analogously. □

A bounded and linear operator T on an ordered Banach space X is said to be *positive* if $T(X^+) \subset X^+$. We express this by $T \geq 0$. If A is a closed linear operator in X we say that A is *resolvent positive* provided there exists $\lambda_0 \geq 0$ such that $[\lambda_0, \infty)$ belongs to the resolvent set $\varrho(-A)$ of $-A$ and $(\lambda + A)^{-1} \geq 0$ for $\lambda \geq \lambda_0$.

Theorem 7.8. *Suppose that* d *satisfies the hypotheses of Corollary 7.5. Then, the evolution operator* U_{A_p} *of* $A_p = A_p[d]$ *is a positive operator on* $L_p[L_\sigma(Y)]$ *for* $1 < p, \sigma < \infty$.

Proof. In view of [**8**, II.Thm.6.4.2, II.Thm.6.4.1] it suffices to prove that for fixed $s \geq 0$ the closed linear operator $B_p := A_p(s)$ in $L_p[L_\sigma(Y)]$ is resolvent positive.
(i) Assume that $\sigma \geq p > n$. From (the proof of) [**4**, Thm.6.1] follows the existence of $\lambda_0 \in \mathbb{R}$ such that for each $y \in Y$ and $\lambda \geq \lambda_0$ we have $w \geq 0$ whenever $w \in W_{p,\mathcal{B}}^2(\Omega)$ satisfies $\big(\lambda - d(s, \cdot, y)\Delta\big)w \geq 0$. Owing to $B_p \in \mathcal{H}\big(H_{p,\mathcal{B}}^2[L_\sigma(Y)], L_p[L_\sigma(Y)]\big)$ we find some $\omega_p > 0$ with $[\omega_p, \infty) \subset \varrho(-B_p)$. Let $\lambda \geq \max\{\omega_p, \lambda_0\} =: \lambda(p)$ and $v \in BUC^\infty(\Omega)^+ \otimes C_c^+(Y)$ be arbitrary so that

$$w := (\lambda + B_p)^{-1} v \in H_{p,\mathcal{B}}^2[L_\sigma(Y)] \hookrightarrow H_{p,\mathcal{B}}^2[L_p(Y)] \ , \tag{7.10}$$

and hence

$$(\lambda + B_p)w = v \geq 0 \quad \text{in} \quad L_p[L_\sigma(Y)] \hookrightarrow L_p[L_p(Y)] \ . \tag{7.11}$$

Recalling the facts that $H_{p,\mathcal{B}}^2[L_p(Y)] \doteq W_{p,\mathcal{B}}^2[L_p(Y)]$, that $L_p[L_p(Y)] = L_p\big(Y, L_p(\Omega)\big)$, and that $BUC^\infty(\Omega) \otimes C_c(Y)$ is dense in $W_p^2[L_p(Y)]$, one obtains from (7.10) and (7.11) that $w(\cdot, y) := w(\cdot)(y)$ belongs for a.a. $y \in Y$ to $W_{p,\mathcal{B}}^2(\Omega)$ and satisfies

$$\big(\lambda - d(s, \cdot, y)\Delta\big)w(\cdot, y) = v(\cdot, y) \geq 0 \ .$$

Therefore, $w(x, y) \geq 0$ for each $x \in \Omega$ and a.a. $y \in Y$ so that $w \in L_p^+[L_\sigma(Y)]$. We conclude

$$(\lambda + B_p)^{-1} v \geq 0 \ , \quad v \in BUC^\infty(\Omega)^+ \otimes C_c^+(Y) \ , \quad \lambda \geq \lambda(p) \ . \tag{7.12}$$

Next use Lemma 7.7 and the closedness of the positive cone $L_p^+[L_\sigma(Y)]$ in $L_p[L_\sigma(Y)]$ to deduce that (7.12) remains valid for $v \in L_p^+[L_\sigma(Y)]$.
(ii) Assume now that $1 < p, \sigma < \infty$ are arbitrary. Choose $\tau \geq \sigma$ and $q \geq p$ such that $\tau \geq q > n$. For $\omega_p > 0$ with $[\omega_p, \infty) \subset \varrho(-B_p)$ put $\omega := \max\{\omega_p, \lambda(q)\}$ where $\lambda(q)$ is given as in (i). According to Lemma 7.7, the space $L_q^+[L_\tau(Y)]$ is dense in $L_p^+[L_\sigma(Y)]$. Regarding this, the assertion follows from (i) since $(\lambda + B_p)^{-1}$ is for any $\lambda \geq \omega$ a bounded and linear operator on $L_p[L_\sigma(Y)]$ satisfying (see the proof of Lemma 7.6)

$$(\lambda + B_p)^{-1}\big|_{L_q[L_\tau(Y)]} = (\lambda + B_q)^{-1} \in \mathcal{L}\big(L_q[L_\tau(Y)]\big) \ .$$

□

Recall that $E = L_2(Y)$, $1 < p < \infty$, and $1 \le \mathsf{p} < \infty$. We denote by $\{S_p^\mu[E]\,;\,\mu > 0\}$ either the scale $\{B_{p,\mathsf{p}}^\mu[E]\,;\,\mu > 0\}$ or the scale $\{H_p^\mu[E]\,;\,\mu > 0\}$. Moreover, we put

$$S_{p,\mathcal{B}}^\mu[E] := \begin{cases} \{u \in S_p^\mu[E]\,;\,\partial_\nu u = 0\}\ , & \mu > 1 + 1/p\ , \\ S_p^\mu[E]\ , & 0 < \mu < 1 + 1/p\ , \end{cases} \tag{7.13}$$

and $S_{p,\mathcal{B}}^\mu[E]^+ := S_{p,\mathcal{B}}^\mu[E] \cap L_p^+[E]$.

Corollary 7.9. *Let $1 < p < \infty$ and suppose that $0 < \mu \le \eta < 2$ with $\mu, \eta \ne 1 + 1/p$. Then $S_{p,\mathcal{B}}^\eta[E]^+$ is dense in $S_{p,\mathcal{B}}^\mu[E]^+$.*

PROOF. Taking into account Theorem 6.30 and (the proof of) Theorem 7.8, this follows from [**8**, V.Prop.2.7.1]. □

7.3. Well-Posedness and Conservation of Mass

After having made available all the tools we need, we can establish now well-posedness of Problem $(**)$. To this end, let us rewrite these equations according to section 7.1 and section 7.2 as a Cauchy Problem of the form

$$\begin{aligned} \dot{u} + A(t)u &= L(t,u)\ , \quad t > 0\ , \\ u(0) &= u^0\ , \end{aligned} \tag{$(CP)_{u^0}$}$$

where $L(t,\cdot)$ and $A(t) := A_p[d(t)] \in \mathcal{H}\big(H_{p,\mathcal{B}}^2[E], L_p[E]\big)$ are given by (7.3) and (7.5), respectively, and where $E = L_2(Y)$. Recall that the scale $\{S_{p,\mathcal{B}}^\mu[E]\,;\,\mu > 0\}$ is defined in (7.13) for $1 < p < \infty$ and $1 \le \mathsf{p} < \infty$ and that $\mathbb{F}$ is given by (7.4).

Then the following fundamental theorem is valid guaranteeing existence and uniqueness of maximal solutions of Problem $(CP)_{u^0}$ in $L_p[E]$ for $1 < p < \infty$.

Theorem 7.10. *Let $r > 0$, $\rho \in (0,1)$, and suppose that*

$$\big[t \mapsto \big(\gamma(t), \beta_c(t), \beta_s(t), K(t), P(t), Q(t)\big)\big] \in C^\rho\big(\mathbb{R}^+, BUC^r[\mathbb{F}]\big)$$

and

$$d \in C^\rho\big(\mathbb{R}^+, C(\bar{\Omega} \times \bar{Y})\big) \quad \text{with} \quad d(t,x,y) > 0\ , \quad (t,x,y) \in \mathbb{R}^+ \times \bar{\Omega} \times \bar{Y}\ .$$

Also suppose $n < 4p$ and $\mu \in (n/2p\,,\,2) \setminus \{1 + 1/p\}$. Then, given any $u^0 \in S_{p,\mathcal{B}}^\mu[E]$, Problem $(CP)_{u^0}$ possesses a unique maximal solution $u := u(\cdot\,;u^0)$ satisfying

$$u \in C\big(J(u^0), S_{p,\mathcal{B}}^\mu[E]\big) \cap C^1\big(\dot{J}(u^0), L_p[E]\big) \cap C\big(\dot{J}(u^0), H_{p,\mathcal{B}}^2[E]\big)\ .$$

The maximal interval of existence $J(u^0)$ is open in $\mathbb{R}^+$. If

$$\sup_{t \in J(u^0) \cap [0,T]} \|u(t)\|_{S_{p,\mathcal{B}}^\mu[E]} < \infty\ , \quad T > 0\ , \tag{7.14}$$

then $J(u^0) = \mathbb{R}^+$.
Moreover, the solution $u(\cdot\,;u^0)$ depends continuously on the initial value u^0 in the following sense: For each $T \in \dot{J}(u^0)$ there exists a neighbourhood U of u^0 in $S_{p,\mathcal{B}}^\mu[E]$ such that $J(v^0) \supset [0,T]$ for $v^0 \in U$ and, as $v^0 \to u^0$ in U,

$$u(\cdot\,;v^0) \to u(\cdot\,;u^0) \quad \text{in} \quad C\big([0,T], S_{p,\mathcal{B}}^\mu[E]\big)\ .$$

PROOF. Set $(\mathbb{E}_0, \mathbb{E}_1) := \big(L_p[E], H^2_{p,\mathcal{B}}[E]\big)$ and put for $\theta \in (0,1)$

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,\mathsf{p}} & \text{if} \quad S^\mu_{p,\mathcal{B}}[E] = B^\mu_{p,\mathsf{p};\mathcal{B}}[E]\ , \\ [\cdot,\cdot]_\theta & \text{if} \quad S^\mu_{p,\mathcal{B}}[E] = H^\mu_{p,\mathcal{B}}[E]\ . \end{cases}$$

Clearly, $\mathbb{E}_1 \overset{d}{\hookrightarrow} \mathbb{E}_0$. Furthermore, Theorem 6.30 entails that

$$\mathbb{E}_\theta := (\mathbb{E}_0, \mathbb{E}_1)_\theta \doteq S^{2\theta}_{p,\mathcal{B}}[E]\ , \quad 2\theta \in (0,2) \setminus \{1+1/p\}\ .$$

Fix $\sigma \in (n/2p\,,\,\mu) \setminus \{1+1/p\}$ and $\tau \in \big(0, \min\{2\sigma - n/p, 1+1/p, r, n/p\}\big)$. Then, due to Proposition 7.2, we have

$$\big[t \mapsto L(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+, C^{1-}_b\big(B^\sigma_{p,\mathsf{p};\mathcal{B}}[E], B^\tau_{p,\mathsf{p};\mathcal{B}}[E]\big)\big)\ .$$

Choose $\varepsilon > 0$ small so that

$$0 < \vartheta_0 := \frac{\tau}{2} - \varepsilon < \vartheta_1 := \frac{\sigma}{2} + \varepsilon < \vartheta_2 := \frac{\mu}{2} < 1\ .$$

According to (6.21), Theorem 6.30, and Corollary 7.5 we deduce

$$\big[t \mapsto \big(A(t), L(t,\cdot)\big)\big] \in C^\rho\big(\mathbb{R}^+, \mathcal{H}(\mathbb{E}_1, \mathbb{E}_0) \times C^{1-}_b(\mathbb{E}_{\vartheta_1}, \mathbb{E}_{\vartheta_0})\big)\ .$$

Since $u^0 \in \mathbb{E}_{\vartheta_2}$, the assertion is now a consequence of [**10**, Thm.5.1]. □

Remarks 7.11. (a) In view of (5.4), (5.9), and Proposition 6.12 we can allow $S^\mu_{p,\mathcal{B}}[E]$ to be $W^\mu_{p,\mathcal{B}}[E]$ in Theorem 7.10.
(b) The solution $u(\cdot;u^0)$ is even more regular than stated in Theorem 7.10. Indeed, it holds

$$u(\cdot;u^0) \in C^{\frac{\mu-\eta}{2}}\big(J(u^0), S^\eta_{p,\mathcal{B}}[E]\big)\ , \quad \eta \in (0,\mu] \setminus \{1+1/p\}\ .$$

This is a consequence of [**8**, II.Thm.5.3.1].
(c) The solution $u(\cdot;u^0) = u^{(\mu)}(\cdot;u^0)$ for $u^0 \in S^\mu_{p,\mathcal{B}}[E]$ is independent of μ in the following sense: If $n/2p < \mu < \bar\mu < 2$ with $\mu, \bar\mu \neq 1+1/p$ and $u^0 \in S^{\bar\mu}_{p,\mathcal{B}}[E] \hookrightarrow S^\mu_{p,\mathcal{B}}[E]$ then $u^{(\bar\mu)}(\cdot;u^0) = u^{(\mu)}(\cdot;u^0)$. This follows immediately from (7.14).

Henceforth, in order to simplify the notation, we will write

$$u(t,x,y) := u(t;u^0)(x)(y)\ , \quad (t,x,y) \in J(u^0) \times \Omega \times Y\ ,$$

for the solution $u = u(\cdot;u^0)$ of Problem $(CP)_{u^0}$, and for convenience we will sometimes suppress any of the variables t, x, and y in a given formula.

The purpose of the next theorem is to provide sufficient conditions for mass conservation. Let d be independent of spatial coordinates meaning that

$$d \in C^\rho\big(\mathbb{R}^+, C(\bar Y)\big) \quad \text{with} \quad d(t,y) > 0\ , \quad (t,y) \in \mathbb{R}^+ \times \bar Y\ . \tag{7.15}$$

Further suppose that both scattering and shattering are mass-preserving. More precisely, assume that for each $(t,x) \in \mathbb{R}^+ \times \Omega$ it holds

$$\int_0^{y_0} y''\beta_s(t,x;y+y',y'')\,dy'' = y+y'\ , \quad \text{a.a. } y_0 < y+y' \le 2y_0\ , \tag{7.16}$$

and

$$Q(t,x;y,y')\Big[\int_0^{y+y'} y''\beta_c(t,x;y+y',y'')\,dy'' - y - y'\Big] = 0\ , \quad \text{a.a. } 0 < y+y' \le y_0\ . \tag{7.17}$$

Note that (7.17) and our assumption on $\beta_c(t,x;\cdot,\cdot)$ to belong to $F_{break} = L_\infty\big(Y, L_2(Y)\big)$ for $(t,x) \in \mathbb{R}^+ \times \Omega$ restrict the physical scope of applications since together they imply that collisions of small droplets cannot result in shattering. Indeed, these assumptions entail the existence of a function $z : \mathbb{R}^+ \times \Omega \to \dot{\mathbb{R}}^+$ with

$$Q(t,x;y,y') = 0 \ , \quad \text{a.a. } 0 < y + y' \le z(t,x) \ , \quad (t,x) \in \mathbb{R}^+ \times \Omega \ .$$

This is a consequence of Hölder's inequality.

Theorem 7.12. *Presuppose the hypotheses of Theorem 7.10 and let in addition (7.15)-(7.17) be valid. Then, for each $u^0 \in S^\mu_{p,\mathcal{B}}[E]$, the solution $u(\cdot;u^0)$ conserves the total mass, that is,*

$$\int_\Omega \int_Y yu(t;u^0)\ dydx = \int_\Omega \int_Y yu^0\ dydx \ , \quad t \in J(u^0) \ .$$

Proof. Since Ω and Y are bounded, the map R, defined as

$$R(t) := \int_\Omega \int_Y yu(t)\ dydx \ , \quad t \in J(u^0) \ ,$$

belongs to $C^1\big(\dot{J}(u^0)\big) \cap C\big(J(u^0)\big)$ due to Theorem 7.10. Approximating $u(t) \in W^2_p[E]$ by functions belonging to $BUC^\infty(\Omega) \otimes C_c(Y)$, we obtain the equality

$$\int_\Omega \int_Y yA(t)u(t)\ dydx = -\int_{\partial\Omega} \int_Y yd(t,y)\partial_\nu u(t)\ dyd\sigma(x) = 0 \ , \quad t \in \dot{J}(u^0) \ .$$

Finally, Proposition 7.2 entails $L\big(t,u(t)\big) \in L_1[L_1(Y)]$, $t \in \dot{J}(u^0)$, so that

$$\int_\Omega \int_Y yL\big(t,u(t)\big)\ dydx = 0 \ , \quad t \in \dot{J}(u^0) \ ,$$

is implied by assumptions (7.16), (7.17), and Lemma 2.7. Consequently, we have $\dot{R}(t) = 0$ for each $t \in \dot{J}(u^0)$. □

7.4. Positivity

In order to prove that the solution $u(\cdot;u^0)$ of Problem $(CP)_{u^0}$ is positive for positive initial values u^0, recall that the space $\mathbb{F}$ being defined by (7.4) is an ordered Banach space with positive cone

$$\mathbb{F}^+ := F^+_{break} \times F^+_{break} \times F^+_{scat} \times F^+_{coal} \times F^+_{coal} \times F^+_{coal} \ ,$$

since the spaces F_{break}, F_{scat}, and F_{coal} are themselves ordered Banach spaces.

Theorem 7.13. *In addition to the assumptions of Theorem 7.10 suppose that*

$$\big(\gamma(t), \beta_c(t), \beta_s(t), K(t), P(t), Q(t)\big) \in BUC^r[\mathbb{F}]^+ \ , \quad t \ge 0 \ . \tag{7.18}$$

Then $u^0 \in S^\mu_{p,\mathcal{B}}[E]^+$ implies $u(t;u^0) \in S^\mu_{p,\mathcal{B}}[E]^+$, $t \in J(u^0)$.

Proof. (i) Assume that $n < 2p$ and $\mu \in (n/p\,,\,2) \setminus \{1 + 1/p\}$. In this case, it follows from (5.1), (5.3), (5.5), (6.21), Proposition 6.12, and Theorem 6.30 that the embedding $S^\mu_{p,\mathcal{B}}[E] \hookrightarrow BUC[E]$ is valid. Let $u^0 \in S^\mu_{p,\mathcal{B}}[E]^+$ be arbitrary and choose $T_0 \in \dot{J}(u^0)$. Then Theorem 7.10 implies

$$\|u(t)\|_{BUC[E]} \le c < \infty \ , \quad 0 \le t \le T_0 \ .$$

Hence, the embedding $E \hookrightarrow L_1(Y)$ yields $\omega := \omega(T_0) > 0$ such that

$$\Big| \int_0^{y_0 - y} K(t,x;y,y')\big[P(t,x;y,y') + Q(t,x;y,y')\big] u(t,x,y')\ dy' \Big| \le \frac{\omega}{3}$$

and

$$\Big| \int_{y_0 - y}^{y_0} K(t,x;y,y') u(t,x,y')\ dy' \Big| \le \frac{\omega}{3}$$

for all $t \in [0,T_0]$, $x \in \Omega$, and a.a. $y \in Y$. We can also assume that

$$\Big| \int_0^y \frac{y'}{y} \gamma(t,x;y,y')\ dy' \Big| \le \frac{\omega}{3}\ , \quad 0 \le t \le T_0\ , \quad x \in \Omega\ , \quad \text{a.a. } y \in Y\ .$$

Putting then for $0 \le t \le T_0$ and $v \in C[E]$

$$\begin{aligned} G(t,v) :=& L_b(t,v) + L_c^1(t,v,v) + L_c^2(t,v,v) + L_s^1(t,v,v) \\ & - L_c^3\big(t,v,u(t)\big) - L_s^2\big(t,v,u(t)\big) + \omega v\ , \end{aligned}$$

where the operators $L_h^j(t,\cdot,\cdot)$ are defined as in section 7.1, it follows

$$G\big(t,v(t)\big) \ge 0\ , \quad 0 \le t \le T \le T_0\ , \quad v \in C\big([0,T], C^+[E]\big)\ . \tag{7.19}$$

Moreover, since

$$G\big(t,u(t)\big) = L\big(t,u(t)\big) + \omega u(t)\ , \quad 0 \le t \le T_0\ ,$$

we see that u is a solution of

$$\dot v + B(t)v = G(t,v)\ , \quad 0 < t \le T_0\ , \qquad v(0) = u^0$$

in $L_p[E]$, where

$$B := \omega + A \in C^\rho\big(\mathbb{R}^+, \mathcal{H}\big(H^2_{p,\mathcal{B}}[E], L_p[E]\big)\big)\ .$$

Denote by U_B the evolution operator of B. Choosing $M > 0$ and $T \in (0,T_0]$ appropriately, one proves on the basis of (a slight modification of) Proposition 7.2 and [**8**, II.Lem.5.1.3] that u is the unique fixed point in

$$\mathcal{V}_T := \big\{ v \in C\big([0,T], S^\mu_{p,\mathcal{B}}[E]\big)\ ;\ \|v(t)\|_{S^\mu_{p,\mathcal{B}}[E]} \le M \big\}$$

of the contraction $\Phi : \mathcal{V}_T \to \mathcal{V}_T$ being given by

$$\Phi(v)(t) := U_B(t,0)u^0 + \int_0^t U_B(t,s) G\big(s,v(s)\big)\ ds\ , \quad 0 \le t \le T\ , \quad v \in \mathcal{V}_T\ .$$

Defining then $u_0 := u^0 \in \mathcal{V}_T$ and $u_{k+1} := \Phi(u_k) \in \mathcal{V}_T$ for $k \in \mathbb{N}$, we obtain a sequence which converges towards u in $\mathcal{V}_T$ and satisfies $u_k(t) \ge 0$ for $0 \le t \le T$ and $k \in \mathbb{N}$ in view of (7.19) and Theorem 7.8. Since $L_p^+[E]$ is closed in $L_p[E]$, we conclude $u(t) \ge 0$ for $0 \le t \le T$. Put

$$T^* := \sup\big\{ \tau \in \dot J(u^0)\ ;\ u(t) \ge 0 \text{ for } 0 \le t \le \tau \big\}$$

and assume $T^* < \sup J(u^0)$. Clearly, $u(T^*) \ge 0$ so that a repetition of the above arguments yields a contradiction. Whence $u(t) \ge 0$ for all $t \in J(u^0)$.

(ii) Assume that $n/2p < \mu \le n/p < 2$ with $\mu \ne 1 + 1/p$ and choose $\eta \in (n/p\ ,\ 2) \setminus \{1 + 1/p\}$. Since $S^\eta_{p,\mathcal{B}}[E]^+$ is dense in $S^\mu_{p,\mathcal{B}}[E]^+$ by Corollary 7.9, part (i) and the continuous dependence on the initial value in the sense of Theorem 7.10 (see also Remarks 7.11(c)) entail that

$$u(t;u^0) \ge 0\ , \quad t \in J(u^0)\ , \quad u^0 \in S^\mu_{p,\mathcal{B}}[E]^+\ .$$

Therefore, the statement of the theorem is true in the case $n < 2p$.
(iii) In order to indicate the dependence on p we write for the remainder of the proof

$$u_p := u_p(\cdot; u^0) \in C\big(J_p(u^0), S^\mu_{p,\mathcal{B}}[E]\big)$$

for the solution of Problem $(CP)_{u^0}$ in $L_p[E]$ with initial value $u^0 \in S^\mu_{p,\mathcal{B}}[E]$. In the following, we say that $P(\alpha)$ *is true* for a given $\alpha \in [2,4]$, provided

$$u_p(t;u^0) \geq 0 \ , \quad t \in J_p(u^0) \ , \quad u^0 \in S^\mu_{p,\mathcal{B}}[E]^+ \ ,$$

whenever

$$p \in (1,\infty) \ , \quad n < \alpha p \ , \quad \mu \in (n/2p\, , \, 2) \setminus \{1+1/p\} \ .$$

The goal is then to verify $P(4)$. First we claim that $P(\alpha)$ implies $P(2+\alpha/2)$ for $\alpha \in [2,4)$. To see this, let $\alpha \in [2,4)$ be such that $P(\alpha)$ is true and fix

$$p \in (1,\infty) \quad \text{with} \quad \alpha \leq \frac{n}{p} < 2 + \frac{\alpha}{2} \ . \tag{7.20}$$

We have to show that for $\mu \in (n/2p\, , \, 2) \setminus \{1+1/p\}$

$$u_p(t;u^0) \geq 0 \ , \quad t \in J_p(u^0) \ , \quad u^0 \in S^\mu_{p,\mathcal{B}}[E]^+ \ . \tag{7.21}$$

In a first step we assume that $\max\big\{1+1/p\, , \, n/p - \alpha/2\big\} < \mu < 2$. In this case we can choose $\varepsilon > 0$ small such that $q := n/\alpha + \varepsilon$ and $\sigma := n/2q + \varepsilon$ satisfy $\mu - n/p > \sigma - n/q$. Therefore, (5.3), (5.6), (5.10), and (6.21) combined with Proposition 6.12 entail that

$$S^\mu_{p,\mathcal{B}}[E] \hookrightarrow S^\sigma_{q,\mathcal{B}}[E] \ . \tag{7.22}$$

Now, if $u^0 \in S^\mu_{p,\mathcal{B}}[E]^+$, Theorem 7.10 yields solutions

$$u_p := u_p(\cdot; u^0) \in C\big(J_p(u^0), S^\mu_{p,\mathcal{B}}[E]\big) \tag{7.23}$$

and

$$u_q := u_q(\cdot; u^0) \in C\big(J_q(u^0), S^\sigma_{q,\mathcal{B}}[E]\big) \ ,$$

both satisfying $(CP)_{u^0}$. Moreover, $u_q(t) \geq 0$, $t \in J_q(u^0)$, since $P(\alpha)$ is true. For $\varrho \in \{p,q\}$ denote by U_ϱ the evolution operator of

$$A_\varrho = A_\varrho[d] \in C^\rho\big(\mathbb{R}^+, \mathcal{H}\big(H^2_{\varrho,\mathcal{B}}[E], L_\varrho[E]\big)\big) \ ,$$

so that

$$u_\varrho(t) = U_\varrho(t,0)u^0 + \int_0^t U_\varrho(t,s)L\big(s,u_\varrho(s)\big)\, ds \ , \quad t \in J_\varrho(u^0) \ , \quad \varrho \in \{p,q\} \ .$$

Put $(\mathbb{E}_0, \mathbb{E}_1) := \big(L_q[E], H^2_{q,\mathcal{B}}[E]\big)$ and $\mathbb{E}_\theta := (\mathbb{E}_0, \mathbb{E}_1)_\theta$ for $\theta \in (0,1)$, where

$$(\cdot,\cdot)_\theta := \begin{cases} (\cdot,\cdot)_{\theta,q} & \text{if} \quad S^\mu_{p,\mathcal{B}}[E] = B^\mu_{p,\mathsf{p};\mathcal{B}}[E] \ , \\ [\cdot,\cdot]_\theta & \text{if} \quad S^\mu_{p,\mathcal{B}}[E] = H^\mu_{p,\mathcal{B}}[E] \ . \end{cases}$$

Due to Theorem 6.30 we have $\mathbb{E}_\theta \doteq S^{2\theta}_{q,\mathcal{B}}[E]$ provided $2\theta \in (0,2) \setminus \{1+1/q\}$. In particular, in view of Proposition 7.2 we can choose $\vartheta > 0$ small enough such that for $\theta := \sigma/2 > n/4q$

$$\big[t \mapsto L(t,\cdot)\big] \in C^\rho\big(\mathbb{R}^+, C^{1-}_b(\mathbb{E}_\theta, \mathbb{E}_\vartheta)\big) \ . \tag{7.24}$$

From Lemma 7.6, (7.22), and (7.23) we deduce that $u_p \in C\big(J_p(u^0), \mathbb{E}_\theta\big)$ solves

$$u_p(t) = U_q(t,0)u^0 + \int_0^t U_q(t,s)L\big(s,u_p(s)\big)\, ds \ , \quad t \in J_p(u^0) \ .$$

Since by [**8**, II.Lem.5.1.3]

$$\|U_q(t,s)\|_{\mathcal{L}(\mathbb{E}_\vartheta,\mathbb{E}_\theta)} \le c(T)(t-s)^{\vartheta-\theta} \ , \quad 0 \le s < t \le T \ ,$$

(7.24) entails

$$\|u_p(t)-u_q(t)\|_{\mathbb{E}_\theta} \le c(T)\int_0^t (t-s)^{\vartheta-\theta}\|u_p(s)-u_q(s)\|_{\mathbb{E}_\theta}\, ds$$

for all $0 \le t \le T \in \dot{J}_p(u^0)\cap\dot{J}_q(u^0)$ and thus $u_p(t)=u_q(t)\ge 0$, $t\in J_p(u^0)\cap J_q(u^0)$, by invoking Gronwall's inequality. From this we conclude $u_p(t)\ge 0$, $t\in J_p(u^0)$, by a contradiction argument as in the last step of (i).
Now, if $\mu\in(n/2p\,,\,2)\setminus\{1+1/p\}$ is arbitrary whereas p still satisfies (7.20), we deduce (7.21) from the previous consideration by a density argument as in (ii). Therefore, $P(\alpha)$ indeed implies $P(2+\alpha/2)$.
(iv) Finally, for $j\in\mathbb{N}$ put $\alpha_j := 4-2^{1-j}\nearrow 4$. Owing to (i) and (ii), $P(\alpha_0)$ is true. Applying (iii), we inductively obtain that also $P(\alpha_j)$ is true for $j\ge 1$. Obviously, this proves the theorem. □

7.5. Global Existence

Up to now, we have established under physical reasonable assumptions that Problem $(**)$, that is, the Cauchy Problem $(CP)_{u^0}$, admits — at least local in time — for each non-negative initial value a unique solution which is non-negative and mass-preserving. Of course, one of the questions which still remains concerns global existence. The following theorem and its corollary provide sufficient conditions for global existence of solutions even though these conditions are far from meeting physical or mathematical requirements completely.
We need some estimates on the kernels reading as

$$\beta_c(t,x;y,y') \le b(t,x,y') \ , \quad (t,x)\in\mathbb{R}^+\times\Omega \ , \quad \text{a.a. } y,y'\in Y \ , \tag{7.25}$$

$$\beta_s(t,x;y,y') \le b(t,x,y') \ , \quad (t,x)\in\mathbb{R}^+\times\Omega \ , \quad \text{a.a. } y\in(y_0,2y_0] \ , \quad \text{a.a. } y'\in Y \ , \tag{7.26}$$

and

$$K(t,x;y,y') \le K^*(t)yy' \ , \quad (t,x)\in\mathbb{R}^+\times\Omega \ , \quad \text{a.a. } y,y'\in Y \ , \tag{7.27}$$

where $K^*\in C(\mathbb{R}^+)$ and $b\in C\big(\mathbb{R}^+,L_p[E]\big)$.

Theorem 7.14. *In addition to the hypotheses of Theorem 7.10 suppose that (7.18) and (7.25)-(7.27) are satisfied. Let $u^0\in S^\mu_{p,\mathcal{B}}[E]^+$ and assume that for each $T>0$*

$$\int_Y y u(s,x,y)\, dy \le c(T) < \infty \ , \quad a.a.\ x\in\Omega \ , \quad s\in J(u^0)\cap[0,T] \ . \tag{7.28}$$

Then the solution $u=u(\cdot;u^0)$ exists globally, i.e., $J(u^0)=\mathbb{R}^+$.

Proof. Let $T>0$ be arbitrary and set $J_T := J(u^0)\cap[0,T]$. We write $|\cdot|_E$ for the norm in $E=L_2(Y)$. Temporarily, fix $s\in J_T$ and $x\in\Omega$ such that $u(s,x,\cdot)\in E^+$ and such that (7.28) holds. Invoking Jensen's inequality and Fubini's theorem, we obtain

$$\begin{aligned}|L_b\big(s,u(s)\big)(x)|_E^2 \le 2\int_0^{y_0}\Big(\int_y^{y_0}\gamma(s,x;y',y)u(s,x,y')\,dy'\Big)^2\,dy\\ +2\int_0^{y_0}|u(s,x,y)|^2\Big(\int_0^y\frac{y'}{y}\gamma(s,x;y,y')\,dy'\Big)^2\,dy\\ \le c\int_0^{y_0}\int_0^y|\gamma(s,x;y,y')|^2\,dy'\,|u(s,x,y)|^2\,dy\\ \le c\|\gamma\|^2_{C(J_T,BUC[F_{break}])}|u(s,x)|^2_E\ .\end{aligned}$$

Therefore,

$$|L_b\big(s,u(s)\big)(x)|_E \le c(T)|u(s,x)|_E\ . \tag{7.29}$$

For $v\in E$ we define $\mathcal{C}v\in L_2(\mathbb{R})\cap L_1(\mathbb{R})$ by

$$(\mathcal{C}v)(y) := \begin{cases} yv(y)\ , & y\in Y\ ,\\ 0\ , & else\ ,\end{cases}$$

so that

$$\mathcal{C}v*\mathcal{C}v(y)=\int_0^y(y-y')v(y-y')y'v(y')\,dy'\ ,\quad y\in Y\ .$$

Let the operators $L_h^j(t,\cdot,\cdot)$ be defined as in section 7.1 and put

$$L_h^j(t,v):=L_h^j(t,v,v)\ ,\quad t\ge 0\ ,\quad v\in E^\Omega\ .$$

Then we deduce in view of (7.27)

$$\begin{aligned}|L_c^1\big(s,u(s)\big)(x)|_E^2 &\le c(T)\int_0^{y_0}\Big(\int_0^y K(s,x;y',y-y')u(s,x,y')u(s,x,y-y')\,dy'\Big)^2\,dy\\ &\le c(T)\,|\mathcal{C}u(s,x)*\mathcal{C}u(s,x)|_E^2\\ &\le c(T)\,|\mathcal{C}u(s,x)|^2_{L_1(\mathbb{R})}|\mathcal{C}u(s,x)|^2_{L_2(\mathbb{R})}\\ &\le c(T)\,|u(s,x)|_E^2\ ,\end{aligned}$$

where we additionally used Young's inequality for convolutions and (7.28). This entails

$$|L_c^1\big(s,u(s)\big)(x)|_E \le c(T)\,|u(s,x)|_E\ . \tag{7.30}$$

Next use Fubini's theorem, (7.25), (7.27), the positivity of $u(s,x)$, and (7.28) to conclude

$$\begin{aligned}|L_c^2\big(s,u(s)\big)(x)|_E^2 = \frac14\int_0^{y_0}\Big(\int_y^{y_0}\int_0^{y'}K(s,x;y'',y'-y'')Q(s,x;y'',y'-y'')\\ \beta_c(s,x;y',y)u(s,x,y'')u(s,x,y'-y'')\,dy''dy'\Big)^2\,dy\\ \le c(T)\int_0^{y_0}|b(s,x,y)|^2\Big(\int_0^{y_0}\int_0^{y_0}y''y'u(s,x,y'')u(s,x,y')\,dy''dy'\Big)^2\,dy\\ \le c(T)\,|b(s,x)|_E^2\end{aligned}$$

and whence

$$|L_c^2\big(s,u(s)\big)(x)|_E \le c(T)\,|b(s,x)|_E\ . \tag{7.31}$$

Similar arguments lead to

$$|L_s^1\big(s,u(s)\big)(x)|_E \le c(T)\,|b(s,x)|_E\ . \tag{7.32}$$

Finally, the estimate

$$|L_c^3\big(s,u(s)\big)(x)|_E + |L_s^2\big(s,u(s)\big)(x)|_E \le c(T)\,|u(s,x)|_E \tag{7.33}$$

is easily obtained from (7.27) and (7.28). Consequently, (7.29)-(7.33) yield

$$\|L\big(s,u(s)\big)\|_{L_p[E]} \le c(T)\,\|u(s)\|_{L_p[E]} + c(T)\ , \quad s\in J_T\ ,$$

since $b \in C\big(\mathbb{R}^+, L_p[E]\big)$. Denoting by U_A the evolution operator of A on $L_p[E]$ (see Corollary 7.5) and taking into account that [**8**, II.Lem.5.1.3] combined with Theorem 6.30 entail

$$\|U_A(t,s)\|_{\mathcal{L}(L_p[E],S^\mu_{p,\mathcal{B}}[E])} \le c(T)(t-s)^{-\mu/2}\ , \quad 0\le s<t\le T\ ,$$

we conclude that for $t\in J_T$

$$\begin{aligned}\|u(t)\|_{S^\mu_{p,\mathcal{B}}[E]} &\le \|U_A(t,0)u^0\|_{S^\mu_{p,\mathcal{B}}[E]} + \int_0^t \|U_A(t,s)\|_{\mathcal{L}(L_p[E],S^\mu_{p,\mathcal{B}}[E])}\|L\big(s,u(s)\big)\|_{L_p[E]}\ ds\\ &\le c(T,u^0) + c(T)\int_0^t (t-s)^{-\mu/2}\|u(s)\|_{L_p[E]}\ ds\ .\end{aligned}$$

The embedding $S^\mu_{p,\mathcal{B}}[E] \hookrightarrow L_p[E]$ and Gronwall's inequality imply then (7.14). □

The next corollary entails the a priori estimate (7.28) — and hence global existence — in the case when the total mass is conserved and when the diffusion operator $-d\Delta$ does not depend on any of the variables t, x, and y.

Corollary 7.15. *In addition to the hypotheses of Theorem 7.10 suppose that $d>0$ is a constant. Further suppose that (7.16)-(7.18) and (7.25)-(7.27) hold. Then, for any $u^0\in S^\mu_{p,\mathcal{B}}[E]^+$ satisfying*

$$\operatorname*{ess-sup}_{x\in\Omega}\int_Y yu^0(x,y)\ dy < \infty\ , \tag{7.34}$$

the solution $u=u(\cdot;u^0)$ exists globally.

Proof. We already know that

$$u = u(\cdot;u^0) \in C\big(J(u^0), S^\mu_{p,\mathcal{B}}[E]\big)\cap C^1\big(\dot{J}(u^0), L_p[E]\big)\cap C\big(\dot{J}(u^0), H^2_{p,\mathcal{B}}[E]\big)$$

is a non-negative and mass-preserving solution of $(CP)_{u^0}$. Putting

$$w(s,x) := \int_Y yu(s,x,y)\ dy\ , \quad s\in J(u^0)\ , \quad \text{a.a. } x\in\Omega\ ,$$

and taking into account (7.16), (7.17), and Lemma 2.7, one shows (using the facts that the smooth $C_c(Y)$-valued functions on $\bar\Omega$ form a dense subspace of $H^2_q[L_q(Y)] \doteq W^2_q[L_q(Y)]$ and that $L_q[L_q(Y)] = L_q\big(Y, L_q(\Omega)\big)$) that w solves

$$\dot w - d\Delta w = 0\ , \quad \partial_\nu w = 0\ ,$$

in $L_q(\Omega)$, where $q:=\min\{p,2\}$. Since the scalar-valued Laplace operator with respect to Neumann boundary conditions generates a (not strongly continuous) semigroup of contractions on $L_\infty(\Omega)$ (see [**52**]), we conclude

$$\|w(s,\cdot)\|_{L_\infty(\Omega)} \le \|w(0,\cdot)\|_{L_\infty(\Omega)}\ , \quad s\in J(u^0)\ ,$$

and thus (7.28). □

Remark 7.16. Observe that

$$S^{\mu}_{p,\mathcal{B}}[E] \hookrightarrow BUC[E] \hookrightarrow L_\infty[L_1(Y)] \ , \quad n/p < \mu < 2 \ ,$$

so that (7.34) is redundant for $u^0 \in S^{\mu}_{p,\mathcal{B}}[E]^+$.

Examples 7.17. Let us consider some examples of kernels in order to illustrate our results. To keep things simple, we omit time dependence as well as dependence on spatial coordinates. Throughout suppose that the shattering probability Q is zero for small droplets, that is, for the symmetric function $Q \in L^+_\infty(Y \times Y)$ there exists some $z_0 \in (0, y_0]$ such that $Q(y, y') = 0$ for a.a. $0 < y + y' \leq z_0$.

(I) As in Examples 2.21(III) assume that fragmentation is subject to a *power-law breakup* meaning that the kernels are of the form

$$\begin{aligned} \gamma(y, y') &:= h y^{\alpha-\xi-1}(y')^{\xi} \ , && 0 < y' < y \leq y_0 \ , \\ \beta_c(y, y') &:= (\zeta + 2) y^{-1-\zeta}(y')^{\zeta} \ , && z_0 < y \leq y_0 \ , \quad 0 < y' < y \ , \\ \beta_s(y, y') &:= (\nu + 2) y_0^{-2-\nu} y (y')^{\nu} \ , && 0 < y' \leq y_0 < y \leq 2y_0 \ , \end{aligned}$$

with $\alpha \geq 1/2\,,\ 0 \geq \xi, \zeta, \nu > -1/2$, and $h > 0$. Extending γ and β_c by zero it is easily seen that

$$(\gamma, \beta_c, \beta_s) \in F^+_{break} \times F^+_{break} \times F^+_{scat} \ .$$

Moreover, (7.16), (7.17) as well as (7.25) and (7.26) are satisfied. Coalescence kernels obeying (7.27) are for instance those of the form

$$K(y, y') = K^*(yy')^{\sigma} \ , \quad y, y' \in Y \ ,$$

where $K^* > 0$ and $\sigma \geq 1$.

(II) Also fragmentation governed by a *parabolic breakup* can be considered as done in Examples 2.21(IV). Given $h > 0$ and $\alpha + \xi + \zeta \geq -1/2$ with $\xi, \zeta > -1/2$, put

$$\gamma(y, y') := h y^{\alpha}(y')^{\xi}(y - y')^{\zeta} \ , \quad 0 < y' < y \leq y_0 \ ,$$

and extend γ by zero so that

$$\int_0^{y_0} |\gamma(y, y')|^2 \, dy' \leq h^2 \mathsf{B}(2\xi + 1, 2\zeta + 1) y_0^{2(\alpha+\xi+\zeta)+1} \ , \quad y \in Y \ ,$$

where B denotes the beta function. Therefore, $\gamma \in F^+_{break}$. For $\sigma \geq \nu > -1/2$ define

$$\beta_c(y, y') := \big(\mathsf{B}(\nu + 2, \sigma + 1)\big)^{-1} y^{-1-\nu-\sigma}(y')^{\nu}(y - y')^{\sigma} \ , \quad z_0 < y \leq y_0 \ , \quad 0 < y' < y \ ,$$

and extend it again by zero. Observe that σ is chosen larger than ν in order to guarantee that the total number of daughter droplets resulting from shattering is at least 2 [1], that is,

$$\int_0^y \beta_c(y, y') \, dy' = \frac{\nu + \sigma + 2}{\nu + 1} \geq 2 \ , \quad z_0 < y \leq y_0 \ .$$

We have

$$\int_0^{y_0} |\beta_c(y, y')|^2 \, dy' \leq \frac{\mathsf{B}(2\nu + 1, 2\sigma + 1)}{\mathsf{B}(\nu + 2, \sigma + 1)} z_0^{-1} \ , \quad z_0 < y \leq y_0 \ ,$$

and whence $\beta_c \in F^+_{break}$ satisfies (7.17). In addition, if $\sigma \geq 0$ then

$$\beta_c(y, y') \leq \frac{z_0^{-1-\nu-\sigma}}{\mathsf{B}(\nu + 2, \sigma + 1)} (y')^{\nu}(y_0 - y')^{\sigma} =: b(y') \ , \quad 0 < y' < y \leq y_0 \ ,$$

[1] For similar reasons we chose ξ, ζ and ν smaller than 0 in (I).

with $b \in E = L_2(Y)$ so that (7.25) holds. Finally, set

$$\beta_s(y,y') := f(y)(y')^{\delta}(y-y')^{\omega} \ , \quad 0 < y' \le y_0 < y \le 2y_0 \ ,$$

for $\omega \ge \delta > -1/2$, where

$$f(y) := y\Big(\int_0^{y_0} z^{1+\delta}(y-z)^{\omega}\ dz\Big)^{-1} \ .$$

Then $\beta_s \in F^+_{scat}$ fulfills (7.16) and (7.26).

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